Chapter 5

The Determinant

5.1 Definition of the Determinant

Given a $n \times n$ matrix A, we would like to define its determinant. We already have a definition for the 2×2 matrix. We define the determinant of a $n \times n$ matrix recursively.

For a $n \times n$ matrix A, define A_{ij} to be the matrix that is obtained by striking out the ith row and j-th column. This is $(n-1) \times (n-1)$ matrix is called the *ij minor of A*. We define the determinant by:

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{13} + \cdots + (-1)^{n+1}a_{1n}\det A_{1n} \tag{5.1.1}$$

For a 1×1 matrix, that is to say, a number, we define:

$$\det(a) = a. \tag{5.1.2}$$

Since the minors A_{ij} are $(n-1) \times (n-1)$ matrices, the $n \times n$ determinant A is determined in terms of matrices of smaller matrices. For example, the determinant of the 2×2 matrix may be written as:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\det d - b\det c = ad - bc. \tag{5.1.3}$$

The determinant of the 3×3 matrix is as follows:

$$\begin{aligned} \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &= a\det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det \begin{pmatrix} d & e \\ g & h \end{pmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg). \end{aligned} \tag{5.1.4}$$

The above definition of the determinant is clear, but is difficult to understand.

Recall that our intuition of the determinant was that it was the signed area of a parallelogram spanned by the two column vectors. The three-dimensional determinant should thus be the volume of a parallelepiped spanned by its three column vectors, and indeed it is. Given a matrix 3×3 matrix A whose column vectors are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ (in this order), the determinant of A is in fact equal to:

$$\det A = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \tag{5.1.5}$$

where $\cdot$ is the dot product and $\times$ is the cross product of vectors. The determinant of an $n \times n$ matrix should thus be defined as the n-dimensional volume of the parallelepiped spanned by the n column vectors of the matrix. To show that this is indeed the case, we show that the determinant satisfies a number of properties that an n-dimensional volume should satisfy.

Proposition 9.

$$\det(I) = 1, \tag{5.1.6}$$

where I is the identity matrix.

In the 2 and 3 dimensional cases, this says that the unit square and unit cube have volume 1. The above is thus a generalization of this property to n dimension.

Proof of Proposition 9. This can be proved by induction on the size of the matrix n. Let I_n be the $n \times n$ identity matrix. When $n = 1$, using (5.1.1), we have:

$$det(I_1) = \det(1) = 1. \tag{5.1.7}$$

Suppose (5.1.6) is true for the $(n-1) \times (n-1)$ identity matrix. We must show that (5.1.6) is true for the $n \times n$ identity matrix. Using (5.1.1), we have:

$$\det(I_n) = 1 \times \det(I_{n-1}), \tag{5.1.8}$$

since only the first term in (5.1.1) is not zero. By our induction hypothesis (that is to say, $\det I_{n-1} = 1$), we have $\det(I_n) = 1$. □

Let a matrix A be a $n \times n$ matrix with column vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$. We will write

$$\det(A) = \det(\mathbf{v}_1, \cdots, \mathbf{v}_n). \tag{5.1.9}$$

Proposition 10. *If we multiply one of the columns of a matrix by a scalar c, the determinant is multiplied by the same scalar. That is to say,*

$$\det(\cdots, c\mathbf{v}_i, \cdots) = c\det(\cdots, \mathbf{v}_i, \cdots). \tag{5.1.10}$$

Geometrically, this says that the volume of the n-dimensional parallelepiped should be multiplied by c if one of its sides is lengthened by a factor of c.

Proof of Proposition 10. We prove this by induction on the size of the matrix n. The case $n = 1$ is clear. Suppose this is true for $n - 1$. We want to prove this is true for n. Let $A = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$ where $\mathbf{v}_i$ are the column vectors. Let $B = (\mathbf{v_1}, \cdots, c\mathbf{v}_i, \cdots, \mathbf{v}_n)$. Using (5.1.1), we have

$$\det(B) = b_{11}\det(B_{11}) - \cdots + (-1)^{k+1} b_{1k}\det(B_{1k}) + \cdots + (-1)^{n+1} b_{1n}\det(B_{1n}), \tag{5.1.11}$$

where b_{ij} is the ij element of B and B_{ij} is the ij minor of B. Likewise, let a_{ij} be the ij element of B and let A_{ij} be the ij minor of A. When $k \neq i$, we have:

$$b_{1k} = a_{1k},\ \det(B_{1k}) = c\det(A_{1k}). \tag{5.1.12}$$

The second equality comes from the induction hypothesis. When $k = i$, we have:

$$b_{1i} = ca_{1i},\ \det(B_{1k}) = \det(A_{1k}) \tag{5.1.13}$$

since $B_{1k} = A_{1k}$. Thus,

$$b_{1k}\det(B_{1k}) = ca_{1k}\det(A_{1k}) \tag{5.1.14}$$

for all k. This proves $\det(B) = c\det(A)$. □

Proposition 11. *If the i-th column of a matrix is a sum of two vectors $\mathbf{v}_i$ and $\mathbf{w}_i$, then its determinant is equal to the the sum of two determinants of matrices whose i-th column has been replaced by $\mathbf{v}_i$ and $\mathbf{w}_i$.*

$$\det(\cdots, \mathbf{v}_i + \mathbf{w}_i, \cdots) = \det(\cdots, \mathbf{v}_i, \cdots) + \det(\cdots, \mathbf{w}_i, \cdots). \tag{5.1.15}$$

This should also be seen as a generalization of what we know about the area of parallelograms and the volume of parallelepipeds. (Try to understand this statement from a geometric point of view!)

Proof of Proposition 11. We prove this statement by induction on the matrix size n. For $n = 1$, the statement is clear from (5.1.2). Suppose the above is true for $n-1$. Let $A = (\cdots, \mathbf{v}_i, \cdots)$, $B = (\cdots, \mathbf{w}_i, \cdots)$ and $C = A + B = (\cdots, \mathbf{v}_i + \mathbf{w}_i, \cdots)$. We let a_{ij} be the ij component of A and A_{ij} be the ij minor of A. By (5.1.1), we have:

$$\det(C) = c_{11}\det(C_{11}) - \cdots + (-1)^{k+1}(c_{1k})\det(C_{1k}) + \cdots + (-1)^{n+1}c_{1n}\det(C_{1n}). \tag{5.1.16}$$

For $k \neq i$, we have:

$$c_{1k} = a_{1k} = b_{1k},\ \det(C_{1k}) = \det(A_{1k}) + \det(B_{1k}). \tag{5.1.17}$$

The second equality follows from the induction hypothesis. For $k = i$, we have

$$c_{1i} = a_{1i} + b_{1i},\ \det C_{1i} = \det(A_{1i}) = \det(B_{1i}). \tag{5.1.18}$$

The second equality follows since $C_{1i} = A_{1i} = B_{1i}$. Thus, we have:

$$c_{1k}\det(C_{1k}) = a_{1k}\det(A_{1k}) + b_{1k}\det(B_{1k}) \tag{5.1.19}$$

for any k. This shows that $\det(C) = \det(A) + \det(B)$. □

Proposition 12. *If two columns of a matrix are the same, then its determinant is* 0*:*

$$\det(\cdots, \mathbf{v}, \cdots, \mathbf{v}, \cdots) = 0. \tag{5.1.20}$$

Geometrically, this says that if the two column vectors of a matrix are equal, then the volume of the (n-dimensional) parallelogram/prallelepiped spanned by these n-vectors must be 0. Before we prove this result, we prove the following fact, assuming Proposition 12.

Proposition 13. *If two columns of the matrix are interchanged, the determinant acquires a minus sign:*

$$\det(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j, \cdots) = \det(\cdots, \mathbf{v}_j, \cdots, \mathbf{v}_i, \cdots). \tag{5.1.21}$$

Proof. Given Propositon 12, we have:

$$\det(\cdots, \mathbf{v}_i + \mathbf{v}_j, \cdots, \mathbf{v}_i + \mathbf{v}_j, \cdots) = 0. \tag{5.1.22}$$

Now, by Proposition 11, we have:

$$\begin{aligned} &\det(\cdots, \mathbf{v}_i + \mathbf{v}_j, \cdots, \mathbf{v}_i + \mathbf{v}_j, \cdots) \\ =&\det(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_i, \cdots) + \det(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j, \cdots) \\ &+ \det(\cdots, \mathbf{v}_j, \cdots, \mathbf{v}_i, \cdots) + \det(\cdots, \mathbf{v}_j, \cdots, \mathbf{v}_j, \cdots). \end{aligned} \tag{5.1.23}$$

The first and last term after the equality must be 0 by Proposition 12. Therefore, we have:

$$\det(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j, \cdots) + \det(\cdots, \mathbf{v}_j, \cdots, \mathbf{v}_i, \cdots) = 0. \tag{5.1.24}$$

□

Proof of Proposition 12. We prove this by induction on the size of the matrix. For $n = 2$, this is cleary true (note that this statement only makes sense only for $n \geq 2$!). Suppose the statement is true for $n - 1$. We may assume that Proposition 13 is true for $(n-1) \times (n-1)$ matrices (the reader should think about why this is OK). Suppose the i-th and j-th columns of the matrix A $(i < j)$ is equal to the vector $\mathbf{v}$. With our usual notation for components and minors, we have:

$$\det(A) = a_{11}\det(A_{11}) - \cdots + (-1)^{k+1}(a_{1k})\det(A_{1k}) + \cdots + (-1)^{n+1}a_{1n}\det(A_{1n}). \tag{5.1.25}$$

If $k \neq i$ and $k \neq j$, we have $\det A_{1k} = 0$ by the induction hypothesis. We thus have:

$$\det(A) = (-1)^{i+1}a_{1i}\det(A_{1i}) + (-1)^{j+1}a_{1j}\det(A_{1j}). \tag{5.1.26}$$

Since the i-th and j-th columns of A are the same, $a_{1i} = a_{1j}$. We must consider the relation between minor A_{1i} and A_{1j}. If $j = i + 1$, A_{1i} and A_{1j} are the same matrix. Therefore, $\det A_{1i} = \det A_{1j}$. If $j = i + 2$, then A_{1i} can be obtained from A_{1j} by interchanging the i-th column and $i + 1$-th column. Thereofre, $\det A_{1i} = -\det A_{1j}$ by Proposition 13 (which is true for $(n - 1) \times (n - 1)$ matrices by our induction hypothesis). If $j = i + 3$, A_{1i} can be obtained from A_{1j} by interchanging the $i + 2$-th column and the $i + 1$-th column, and then, the $i + 1$-th column and the i-th column. Thus, $\det A_{1i} = (-1)^2 \det A_{1j} = \det A_{1j}$ In general, we need $i - j - 1$ interchanges of columns to obtain A_{1i} from A_{1j}. Therefore,

$$\det A_{1i} = (-1)^{i-j-1}\det A_{1j}. \tag{5.1.27}$$

Thus, (5.1.26) evaluates to

$$\det(A) = (-1)^{i+1+i-j-1}a_{1j}\det(A_{1j}) + (-1)^{j+1}a_{1j}\det(A_{1j}) = 0. \tag{5.1.28}$$

□

We may now prove the following important fact.

Proposition 14. *Suppose the column vectors of the* $n \times n$ *matrix* A *are linearly dependent. Then,* $\det A = 0$.

The above proposition should be natural. If the column vectors are linarly dependent, they lie on a subspace with dimension smaller than n. This means that the the n-dimensional parallelepiped spanned by the n vectors should be 0

Proof of Proposition 14. Let $A = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$. Since the column vectors are linearly dependent, we have:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0 \tag{5.1.29}$$

where not all c_k are equal to 0. Assume that $c_1 \neq 0$ (the proof when some other $c_k \neq 0$ is similar). We have:

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 - \cdots - \frac{c_n}{c_1}\mathbf{v}_n. \tag{5.1.30}$$

Using Proposition 10 and Proposition 11, we have:

$$\det(\mathbf{v}_1, \cdots, \mathbf{v}_n) = \sum_{k=1}^{n} -\frac{c_k}{c_1}\det(\mathbf{v}_k, \cdots, \mathbf{v}_k, \cdots). \tag{5.1.31}$$

By Proposition 12, all terms in the above sum are 0. □

A beautiful fact about determinants is that the properties we just proved of the determinant, can in fact be used to give an alternative definition of the determinant. Consider a function δ that assignes to each $n \times n$ matrices A a scalar $\delta(A)$. Let $A = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$, and we shall write $\delta(A) = \delta(\mathbf{v}_1, \cdots, \mathbf{v}_n)$. Now suppose that δ satisfies the following properties:

$$\delta(I) = 1 \tag{5.1.32}$$

$$\delta(\cdots, c\mathbf{v}_i, \cdots) = c\delta(\cdots, \mathbf{v}_i, \cdots) \tag{5.1.33}$$

$$\delta(\cdots, \mathbf{v}_i + \mathbf{w}_i, \cdots) = \delta(\cdots, \mathbf{v}_i, \cdots) + \delta(\cdots, \mathbf{w}_i, \cdots) \tag{5.1.34}$$

$$\delta(\cdots, \mathbf{v}, \cdots, \mathbf{v}, \cdots) = 0 \tag{5.1.35}$$

Note also that, if δ satisfies the above properties, it must also satisfy the property of Proposition 13 (why is that?):

$$\delta(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j, \cdots) = -\delta(\cdots, \mathbf{v}_j, \cdots, \mathbf{v}_i, \cdots). \tag{5.1.36}$$

We have seen in Propositions 9, 10, 11 and 12 that the determinant satisfies these properties. It turns out that the determinant is the only function that satisfies these properties. Recall that these properties are natural properties that we might expect of the volume function of a n-dimensional parallelepiped. The determinant is thus a natural generalization of the concept of parallelogram area/parallelepiped volume to n-dimensions. We shall prove a theorem that is slightly more general than this.

Theorem 11. *Suppose δ is a function that assigns to each $n \times n$ matrix a scalar value. If δ satisfies* (5.1.33)-(5.1.35) *and*

$$\delta(I) = k \tag{5.1.37}$$

where k is a scalar constant and I is the identity matrix. Then the function δ must be k times the determinant.

As we shall see later, this is a powerful theorem since it says that we have only to check a couple of properties to show that a given function is a determinant.

Proof of Theorem 11. We prove this by induction on the matrix size n. When $n = 2$, we have:

$$\begin{aligned}
\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} =& \delta\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \delta\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \\
=& \delta\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \delta\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \delta\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \delta\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \\
=& ab\delta\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + ad\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bc\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + cd\delta\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\
=& ad\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - bc\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = k(ad - bc).
\end{aligned} \tag{5.1.38}$$

In the first and second equality we used (5.1.34). In the third equality, we used (5.1.33). In the fourth equality, we used (5.1.35) and (5.1.36). In the last equality, we used (5.1.37). Therefore, δ is necessary k times the determinant.

Let us assume the statement has been proved for $n - 1$. We would like to prove the statement for n. Let $A = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$ and let:

$$\mathbf{v}_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}, \; \mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \mathbf{w}_k = \begin{pmatrix} 0 \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{u}_k \end{pmatrix}. \tag{5.1.39}$$

The vectors $\mathbf{u}_k$ are $n-1$ dimensional vectors whose components are equal to the bottom $n-1$ components of $\mathbf{v}_k$ Using $\mathbf{v}_1 = a_{11}\mathbf{e} + \mathbf{w}_1$, we have:

$$\begin{aligned} \delta(\mathbf{v}_1, \cdots, \mathbf{v}_n) &= \delta(a_{11}\mathbf{e}, \mathbf{v}_2, \cdots, \mathbf{v}_n) + \delta(\mathbf{w}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) \\ &= a_{11}\delta(\mathbf{e}, \mathbf{v}_2, \cdots, \mathbf{v}_n) + \delta(\mathbf{w}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) \end{aligned} \tag{5.1.40}$$

where we used (5.1.34) in the first equality and (5.1.33) in the second equality. Now, note that

$$\begin{aligned} \delta(\mathbf{e}, \mathbf{v}_2, \cdots, \mathbf{v}_n) &= a_{12}\delta(\mathbf{e}, \mathbf{e}, \cdots, \mathbf{v}_n) + \delta(\mathbf{e}, \mathbf{w}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n) \\ &= \delta(\mathbf{e}, \mathbf{w}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n), \end{aligned} \tag{5.1.41}$$

where we used (5.1.34) and (5.1.33) in the first equality and (5.1.35) in the second equality. Applying the same procedure to each column, we have:

$$\delta(\mathbf{e}, \mathbf{v}_2, \cdots, \mathbf{v}_n) = \delta(\mathbf{e}, \mathbf{w}_2, \cdots, \mathbf{w}_n) \tag{5.1.42}$$

Now, define the function $\widetilde{\delta}$ assigning $(n-1) \times (n-1)$ matrices to scalars.

$$\widetilde{\delta}(\mathbf{u}_2, \cdots, \mathbf{u}_n) = \delta(\mathbf{e}, \mathbf{w}_2, \cdots, \mathbf{w}_n). \tag{5.1.43}$$

From the fact that δ satisfies properties (5.1.33)-(5.1.35) and (5.1.37) for $n \times n$ matrices, it is easily seen that $\widetilde{\delta}$ satisfies properties (5.1.33)-(5.1.35) and (5.1.37) for $(n-1) \times (n-1)$ matrices. By our induction hypothesis, we see that

$$\widetilde{\delta}(\mathbf{u}_2, \cdots, \mathbf{u}_n) = k \det(\mathbf{u}_2, \cdots, \mathbf{u}_n). \tag{5.1.44}$$

We thus see that (5.1.40) can be written as:

$$\delta(\mathbf{v}_1, \cdots, \mathbf{v}_n) = ka_{11} \det(A_{11}) + \delta(\mathbf{w}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) \tag{5.1.45}$$

We may now perform the same procedure on the second term of the above to see that:

$$\begin{aligned} \delta(\mathbf{w}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) &= a_{12}\delta(\mathbf{w}_1, \mathbf{e}, \mathbf{v}_3 \cdots, \mathbf{v}_n) + \delta(\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n) \\ &= -a_{12}\delta(\mathbf{e}, \mathbf{w}_1, \mathbf{v}_3 \cdots, \mathbf{v}_n) + \delta(\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n) \end{aligned} \tag{5.1.46}$$

where we used (5.1.36) in the second equality. In the same way as before, we see that

$$\delta(\mathbf{e}, \mathbf{w}_1, \mathbf{v}_3 \cdots, \mathbf{v}_n) = k \det(\mathbf{u}_1, \mathbf{u}_3, \cdots, \mathbf{u}_n). \tag{5.1.47}$$

Thus,

$$\delta(\mathbf{v}_1, \cdots, \mathbf{v}_n) = ka_{11} \det(A_{11}) - ka_{12} \det(A_{12}) + \delta(\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n) \tag{5.1.48}$$

Continuing in this way for all columns, we obtain:

$$\begin{aligned}\delta(\mathbf{v}_1,\cdots,\mathbf{v}_n)=&k\left(a_{11}\det(A_{11})-\cdots+(-1)^{n+1}a_{1n}\det(A_{1n})\right)\\&+\delta(\mathbf{w}_1,\cdots,\mathbf{w}_n).\end{aligned}\tag{5.1.49}$$

Now, note that the first component of vectors $\mathbf{w}_1,\cdots,\mathbf{w}_n$ is 0. Thus, the vectors $\mathbf{w}_1,\cdots,\mathbf{w}_n$ all live in a $(n-1)$-dimensional subspace of $\mathbb{R}^n$, and are linearly dependent. For linearly dependent vectors, δ evaluates to 0, by the proof of Proposition 14 (the reader should check this). This completes the proof. □

5.2 Properties of the Determinant

One of the most important properties of the determinant is the product rule.

Theorem 12. *Given two $n\times n$ matrices A and B, the determinant of AB is the product of the determinants of A and B:*

$$\det(AB)=\det(A)\det(B).\tag{5.2.1}$$

Proof. Let $B=(\mathbf{v}_1,\cdots,\mathbf{v}_n)$. We have:

$$\det(AB)=\det(A\mathbf{v}_1,\cdots,A\mathbf{v}_n).\tag{5.2.2}$$

Let

$$\delta(B)=\delta(\mathbf{v}_1,\cdots,\mathbf{v}_n)=\det(A\mathbf{v}_1,\cdots,A\mathbf{v}_n)=\det(AB).\tag{5.2.3}$$

The function δ satisfies the conditions of Theorem 11 where the constant k equal to

$$k=\delta(I)=\det(AI)=\det(A).\tag{5.2.4}$$

Therefore, an application of Theorem 11 yields:

$$\det(AB)=\det(A)\det(B).\tag{5.2.5}$$

□

An immediate consequence of this is the following.

Theorem 13. *A matrix A is invertible if and only if its determinant is not 0.*

Proof. Suppose a given matrix A is invertible. Then, we have:

$$A^{-1}A = I. \tag{5.2.6}$$

By the product rule, we have:

$$\det(A^{-1})\det(A) = \det(I) = 1. \tag{5.2.7}$$

Thus, $\det(A) \neq 0$. If A is not invertible, the column vectors of A are linearly dependent. From Proposition 14, we know that $\det(A) = 0$. □

Another important property to be aware of is the following.

Proposition 15. *The determinants of A and its transpose A^{T} are the same.*

Proof. If $\det(A) = 0$, then the column vectors of A are linearly dependent. This is equivalent to the row vectors of A being linearly dependent (see Theorem 9). Thus, $\det(A^{\mathrm{T}}) = 0$. If $\det(A) \neq 0$, then A is invertible by Theorem 13. Therefore, A can be written as a product of matrices of elementary row operations (see (3.2.10)):

$$A = E_1E_2 \cdots E_N. \tag{5.2.8}$$

Now,

$$A^T = E_N^{\mathrm{T}} \cdots E_1^{\mathrm{T}}. \tag{5.2.9}$$

For matrices E of elementary row operations (check this!),

$$\det(E) = \det(E^{\mathrm{T}}). \tag{5.2.10}$$

Using this and the product rule for the determinant, we see that

$$\det(A) = \det(A^{\mathrm{T}}). \tag{5.2.11}$$

□

Now, we discuss the behavior of the determinant under elementary row and column operations. Elementary column operations are the same as elementary row operations, just that they are applied to columns.

Proposition 16. *The determinant behaves in the following way under elementary row and column operations.*

1. *Multiply the i-th row (column) by a scalar c and add this to the j-th row (column), $i \neq j$. This does not change the value of the determinant.*

2. *Exchange row (column) i with row (column) j, $i \neq j$. The determinant changes sign (is multiplied by -1).*

3. *Multiply the i-th row (column) by a scalar c. The determinant is multiplied by c.*

Proof. We only show that this is true for columns. Indeed, given Proposition 15, we need only prove this for column operations, since performing column operations on a matrix is equivalent to performing row operations on its transpose. The second and last item is just Proposition 13 and Proposition 10 respectively. The first item can be seen as follows. Let the $n \times n$ matrix A consist of column vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$. We see that

$$\begin{aligned}
&\det(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j + c\mathbf{v}_i, \cdots) \\
&= \det(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j, \cdots) + c\det(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_i, \cdots) \\
&= \det(\cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j, \cdots)
\end{aligned} \tag{5.2.12}$$

We used Propositions 10 and 11 in the first equality, and Proposition 12 in the last equality. □

It turns out that the determinant can be expanded not only in the first row, but any row or column.

Proposition 17. *Given a matrix A, let a_{ij} be the components of A and A_{ij} be the ij minor of A. Then,*

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}), \tag{5.2.13}$$

$$\det(A) = (-1)^{i+1} a_{1i} \det(A_{1i}) + \cdots + (-1)^{i+n} \det(A_{ni}), \tag{5.2.14}$$

for any $i, 1 \leq i \leq n$.

An easy way to keep track of the sign is to have in mind the matrix of signs:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.2.15}$$

Proof of Proposition 17. If we can prove (5.2.13), (5.2.14) follows since operations and columns and rows are interchangeable, given Proposition 15.

We prove (5.2.13). When $i = 1$, this is just (5.1.1). If $i \neq 1$, we may interchange rows i and $i-1$, then rows $i-1$ and $i-2$ and so on until rows 2 and 1 are exchanged. Let the resulting matrix be $\widetilde{A}$. Then,

$$\det(A) = (-1)^{i-1}\det(\widetilde{A}). \tag{5.2.16}$$

The first row of $\widetilde{A}$ is given by $(a_{i1}, \cdots, a_{in})$ and $\widetilde{A}_{1k} = A_{ik}$ for all k. Now using (5.1.1), the determinant of $\widetilde{A}$ is:

$$\det(\widetilde{A}) = a_{i1}\det(A_{i1}) - \cdots + (-1)^{n+1}a_{in}\det(A_{in}). \tag{5.2.17}$$

Combining the above with (5.2.16), we obtain (5.2.13). □

The following result is often quite useful.

Proposition 18. *Suppose the $n \times n$ matrix A is written in the following block matrix form:*

$$A = \left(\begin{array}{c|c} P & Q \\ \hline O_{m-n,n} & R \end{array}\right) \tag{5.2.18}$$

where P is a $m \times m$ matrix and $O_{m-n,n}$ is the $(m-n) \times m$ zero matrix. Then,

$$\det(A) = \det(P)\det(R). \tag{5.2.19}$$

The same is true when Q is replaced by the zero matrix and $O_{m-n,n}$ is replaced by an arbitrary matrix.

Proof. If $\det(P) = 0$, then the column vectors of P, and hence A are linearly dependent. Thus, $\det(A) = 0$, and (5.2.19) holds. If $\det(P) \neq 0$, P has an inverse. Therefore, we can write:

$$A = \left(\begin{array}{c|c} P & O_{m,n-m} \\ \hline O_{n-m,m} & I_{n-m} \end{array}\right)\left(\begin{array}{c|c} I_m & P^{-1}Q \\ \hline O_{n-m,m} & R \end{array}\right) \tag{5.2.20}$$

where I_m and I_{n-m} are the $m \times m$ and $(n-m) \times (n-m)$ identity matrices. Now, notice that:

$$\det\left(\begin{array}{c|c} P & O^{\mathrm{T}} \\ \hline O_{n-m,m} & I_{n-m} \end{array}\right) = 1 \cdot \det\left(\begin{array}{c|c} P & O_{m,n-m-1} \\ \hline O_{n-m-1,m} & I_{n-m-1} \end{array}\right) = \cdots = \det(P), \tag{5.2.21}$$

where we have successively expanded the above determinand along the bottom row. In much the same way (this time, successively expanding in terms of minors in the top row), we have:

$$\det\left(\begin{array}{c|c} I_m & P^{-1}Q \\ \hline O_{n-m,m} & R \end{array}\right) = \det(R). \tag{5.2.22}$$

Using (5.2.20), (5.2.21) and (5.2.22) together with the product rule for the determinant, we obtain (5.2.19). The remark in the last line of the statement of the Proposition is true because the determinant of a matrix and its transpose are the same. □

5.3 Evaluating Determinants

As you can see, there were many properties of the determinant, and it takes some practice to get used to all of these properties. Here, we compute some examples.

Example 12. *Let A be an upper triangular matrix:*

$$A = \begin{pmatrix} a_1 & * & \cdots & * \\ 0 & a_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \tag{5.3.1}$$

Then, the determinant of A is:

$$\begin{aligned} \det(A) &= \det \begin{pmatrix} a_1 & * & \cdots & * \\ 0 & a_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \\ &= a_1 \det \begin{pmatrix} a_2 & * & \cdots & * \\ 0 & a_3 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} = \cdots = a_1 a_2 \cdots a_n. \end{aligned} \tag{5.3.2}$$

where we used Proposition 18. The determinant of an upper triangular matrix is therefore just the product along the diagonal. The same is true for lower triangular matrices.

Example 13. *Consider the matrix:*

$$A = \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} \tag{5.3.3}$$

We have:

$$\begin{aligned}\det(A) &= \det\begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} = \det\begin{pmatrix} x+2 & x+2 & x+2 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} \\ &= (x+2)\begin{pmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} = (x+2)\begin{pmatrix} 1 & 1 & 1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{pmatrix} \\ &= (x+2)(x-1)^2. \end{aligned} \tag{5.3.4}$$

(Justify each step!) This says that the determinant of A is equal to 0 if and only if $x = -2$ and $x = 1$. This makes sense. If $x = 1$ or $x = -2$, the column vectors of A are indeed linearly dependent.

Example 14. *Let us compute the determinant of the following 4×4 matrix:*

$$\begin{aligned}&\det\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 & 0 & -1 \\ 2 & 2 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix} \\ &= \det\begin{pmatrix} 2 & 0 & 0 & -1 \\ 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} \cdot \det\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} = 16. \end{aligned} \tag{5.3.5}$$

Again, you should justify each step of this calculation.

5.4 Exercises

1. Compute the determinants of the following matrices.

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}, \begin{pmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-b-a \end{pmatrix}$$
$$\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & -3 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & -3 & 0 \end{pmatrix}, \begin{pmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a & b & c & d \end{pmatrix} \tag{5.4.1}$$

2. Compute the determinants of the matrices of elementary row reduction (Matrices (3.2.3), (3.2.5) and (3.2.7)).

3. Compute the determinant of the following $n \times n$ matrix whose diagonal is x and all other components are 1:

$$A = \begin{pmatrix} x & 1 & 1 & \cdots & 1 \\ 1 & x & 1 & \cdots & 1 \\ 1 & 1 & x & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & x \end{pmatrix} \tag{5.4.2}$$

. (*Hint: Mimic the calculation of Example 13*).

4. Show that if $A^n = O$ where O is the 0 matrix, then $\det A = 0$.

5. Show that the determinant of AA^{T} and $A^{\mathrm{T}}A$ are both non-negative.

6. Given two column vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n, n \geq 2$, form the $n \times n$ matirx $A = \mathbf{a}\mathbf{b}^{\mathrm{T}}$. What is the determinant of A? (*Hint: Examine linear dependence*).