1 Thermal Noise

1.1 History and Background

In 1827 the Scottish botanist Robert Brown observed that pollen grains and other small objects suspended in water perform a random zigzag motion. He initially thought he was "seeing" evidence of life. This motion was given the name *Brownian* for its discoverer. Later in the 19^{th} century it was realized that *Brownian* motion is created by the impact of the molecules of the fluid surrounding the object due to the fluctuations in the ambient thermal energy (i.e. temperature). A theory that explained this phenomena was developed in 1905 by Albert Einstein based on the kinetic theory of gases.

The motion of the particle is essentially a random walk, with steps to the right equally as probable as steps to the left. Einstein suggested that the mean kinetic energy per degree of freedom of the particle should be given by statistical mechanics and the equipartition of energy as

$$\frac{1}{2} M \overline{v_x^2} = \frac{1}{2} k_B T \; ; \tag{1}$$

where M is the mass of the particle, v_x its instantaneous velocity component in the x-direction, $\overline{v_x^2}$ its mean-squared value (the variance for a zero mean process), k_B is Boltzmann's constant, and T the absolute temperature. However, what is actually "observed" under the microscope is not the instantaneous velocity, but the displacement Δx in the x-direction during the time interval t. Einstein showed that

$$\overline{\Delta x^2} = 2\, D\, t \; , \tag{2}$$

where $\overline{\Delta x^2}$ is the mean-squared value of Δx and D is the diffusion constant of the particle. Einstein realized that many physical systems would exhibit *Brownian* motion. Thermal noise in circuits is nothing more than *Brownian* motion of electrons due to the ambient temperature.

The first person to measure Thermal noise in conductors was J. B. Johnson in 1928 while working at Bell Labs. He explained his findings to H. Nyquist who also worked at Bell Labs. Nyquist derived an expression by means of thermodynamics and statistical mechanics which fit Johnson's measurements. The work of these two men has led to Thermal Noise being called Johnson Noise by experimental physicists and Nyquist Noise by theoreticians. Johnson found that thermal agitation of electricity in conductors produces a random variation in the potential between the ends of the conductor. The electromotive force developed across the ends of the conductor due to Thermal noise is unaffected by the presence or absence of direct current. This can be explained by the fact that electron thermal velocities in a conductor are much greater ($\sim 10^3$ times) than electron drift velocities. Johnson also found that the mean-squared voltage fluctuation across the ends of the conductor was directly proportional to the resistance of the conductor and directly proportional to the absolute temperature of the ambient about the conductor.

Thermal noise is a fundamental physical phenomenon and is present in **any** passive resistor above absolute zero temperature. The amplitude distribution of Thermal noise is Gaussian in three dimensions (Central Limit Theorem), which can be illustrated by a random walk process. We will investigate Thermal noise using van der Ziel's method, the mathematical techniques shown in deriving Shot noise and finally Nyquist's derivation.

1.2 van der Ziel's Derivation of Thermal Noise

Consider a resistor R in parallel with a capacitor C. As a result of the random thermal agitation of the electrons in the resistor, the capacitor will be charged and discharged at random. The average energy stored in the capacitor will be:

$$\frac{1}{2}C\,\overline{V^2} = \frac{1}{2}k_BT \ , \qquad \text{or} \qquad \overline{V^2} = \frac{k_BT}{C} \ ; \tag{3}$$

where $\overline{V^2}$ is the mean-square value of the voltage fluctuation empressed across the capacitor. This equation can be proved by the equipartition theorem as follows.

If a system has a temperature T, the probability that it has an energy E is propotional to $\exp(-E/k_BT)$, which is called the Boltzmann factor. In the RC circuit the energy stored in the capacitor C is $\frac{1}{2}CV^2$, where V is the voltage across C. The probability dP of finding a voltage between V and $(V + dV)$ is therefore:

$$dP = K_o \exp\left(-\frac{CV^2}{2k_BT}\right)\ dV \ , \tag{4}$$

where the constant K_o must be chosen so that;

$$\int_{-\infty}^{\infty} K_o \exp\left(-\frac{CV^2}{2k_BT}\right)\ dV = 1 \ . \tag{5}$$

Introducing the new variable

$$x^2 = \frac{CV^2}{2k_BT} \ , \qquad dx = \sqrt{\frac{C}{2k_BT}}\,dV \ ; \tag{6}$$

and then substituting into equation (5) gives

$$K_o \sqrt{\frac{2k_BT}{C}} \underbrace{\int_{-\infty}^{\infty} \boldsymbol{e}^{-x^2} dx}_{\sqrt{\pi}} = 1 \; , \tag{7}$$

therefore

$$K_o = \sqrt{\frac{C}{2\pi k_B T}} \; . \tag{8}$$

Now calculating the mean-squared voltage $\overline{V^2}$ from the second moment gives

$$\overline{V^2} = \sqrt{\frac{C}{2\pi k_B T}} \int_{-\infty}^{\infty} V^2 \exp\left(-\frac{CV^2}{2k_BT}\right) dV \; , \tag{9}$$

introducing the same variable x as before (6) yields

$$\overline{V^2} = \sqrt{\frac{C}{2\pi k_B T}} \sqrt{\left(\frac{2k_BT}{C}\right)^3} \underbrace{\int_{-\infty}^{\infty} x^2 \, \boldsymbol{e}^{-x^2} dx}_{\sqrt{\pi}/2} \; . \tag{10}$$

Finally after simplifying equation (10) gives

$$\overline{V^2} = \frac{k_BT}{C} \; . \tag{11}$$

Now we will prove Nyquist's theorem with the help of a parallel RC circuit. We can model the thermal noise in the resistor as an electromotive force (e.m.f.) in series with a noiseless resistor R. The e.m.f. source has an r.m.s. value of $(S_V(0)\Delta f)^{1/2}$. Solving the RC circuit for the voltage across the capacitor V_c leads to

$$d\overline{V_c^2} = S_V(0) \, df \, \frac{(\omega C)^{-2}}{R^2 + (\omega C)^{-2}} = \frac{S_V(0) df}{1 + (\omega RC)^2} \; . \tag{12}$$

Integrating equation (12) we obtain

$$\overline{V_c^2} = \int_0^{\infty} d\overline{V_c^2} = S_V(0) \int_0^{\infty} \frac{df}{1 + (\omega RC)^2} \; . \tag{13}$$

To solve the integral in equation (13), let $x = \omega RC$, $dx = 2\pi RC \, df$ and substitute into (13).

$$\overline{V_c^2} = \frac{S_V(0)}{2\pi RC} \underbrace{\int_0^{\infty} \frac{dx}{1+x^2}}_{\pi/2} \; . \tag{14}$$

$$\overline{V_c^2} = \frac{S_V(0)}{4RC} \ . \tag{15}$$

Now applying the equipartion law, equation (11), to equation (15) yields

$$\overline{V_c^2} = \frac{S_V(0)}{4RC} = \frac{k_B T}{C} \ . \tag{16}$$

Solving for the spectral density S_V we get

$$S_V(0) = 4k_B T R \ . \tag{17}$$

Equation (17) is called Nyquist's Theorem and the symbol $S_V(0)$ for the spectral density means that there is no frequency dependence. Noise with such a spectrum is called *white*.

1.3 Derivation of Thermal Noise from a Physical Model

We start with the simplest model of current transport through a resistor having a constant scattering rate. Since Thermal noise is an equilibrium phenomenon, we will assume the electric field is zero between the ends of the resistor. From the Shockley-Ramo theorem, the current due to an electron with velocity v crossing a gap ℓ is

$$I(t) = q\mathsf{v}(t)/\ell \ . \tag{18}$$

Then the autocorrelation function for the current due to one electron is

$$\left\langle I(t)I(t+t')\right\rangle = \frac{q^2}{\ell^2}\left\langle \mathsf{v}(t)\mathsf{v}(t+t')\right\rangle \ . \tag{19}$$

To evaluate the average over the velocities, note that with no electric field the velocity of an electron is only changed by a scattering event. The probability that an electron remains unscattered after a time t' is $\boldsymbol{e}^{-t'/\tau}$. After a scattering event, the new velocity is uncorrelated with the old velocity, and consequently makes no contribution to the autocorrelation. Also, we can make the same argument about correlation with prior times as with subsequent times. Thus,

$$\left\langle I(t)I(t+t')\right\rangle = \frac{q^2}{\ell^2}\left\langle \mathsf{v}^2\right\rangle \boldsymbol{e}^{-|t'|/\tau} \ . \tag{20}$$

The expectation value of v^2 is found from an integrated Maxwellian velocity distribution or invoking the equipartition theorem to obtain

$$\left\langle \mathsf{v}^2\right\rangle = k_B T/m^* \ , \tag{21}$$

where m^* is the effective electron mass. So we have

$$\langle I(t)I(t+t')\rangle = \frac{q^2 k_B T}{\ell^2 m^*}\, \boldsymbol{e}^{-|t'|/\tau} \;, \tag{22}$$

for the contribution of each electron to the autocorrelation function $R_I(t')$. Each electron independently contributes to R_I, so we need to multiply by the number of electrons in the resistor; which is $nA\ell$ where n is the electron density, A is the cross-sectional area, and ℓ is the length of the resistor. Therefore, the total current autocorrelation function for Thermal noise is

$$R_I(t') = \frac{q^2 n k_B T A}{m^* \ell}\, \boldsymbol{e}^{-|t'|/\tau} \;. \tag{23}$$

We need to know the Fourier transform pair for (23) which is

$$\mathcal{F}\left\{\boldsymbol{e}^{-|t'|/\tau}\right\} \leftrightarrow \frac{2/\tau}{\omega^2 + 1/\tau^2} \;, \tag{24}$$

where $\omega = 2\pi f$. Now applying the Wiener-Khintchine theorem, we find the noise spectral density to be

$$\begin{aligned} S_I(f) &= \frac{2q^2 n k_B T A}{m^* \ell}\left(\frac{2\tau}{1+\omega^2\tau^2}\right) \\ &= 4k_B T q n \frac{q\tau}{m^*}\frac{A}{\ell}\left(\frac{1}{1+\omega^2\tau^2}\right) \\ &= 4k_B T q n \mu_n \frac{A}{\ell}\left(\frac{1}{1+\omega^2\tau^2}\right) \\ &= 4k_B T \frac{\sigma A}{\ell}\left(\frac{1}{1+\omega^2\tau^2}\right) \\ &= \frac{4k_B T}{R}\left(\frac{1}{1+\omega^2\tau^2}\right) \;. \end{aligned} \tag{25}$$

Here we have used the following definitions from device physics; mobility $\mu_n = q\tau/m^*$, conductivity $\sigma = qn\mu_n$, and resistance $R = \ell/\sigma A$.

This derivation is different from any textbook or technical paper ever written. It was motivated by Prof. Bill Frensley (UTD) who believes that the Fluctuation-Dissipation theorem is violated by the standard development. The frequency dependence seen in equation (25) is called a Lorentzian characteristic. It has a single characteristic time constant τ which sets the corner frequency at $1/2\pi\tau$. A picture of this type of spectrum is shown in Figure 3.

This τ is different from the one used in the Shot noise derivation. The τ here is the mean time between carrier collisions (≈100 fs) with the lattice. This gives the corner frequency

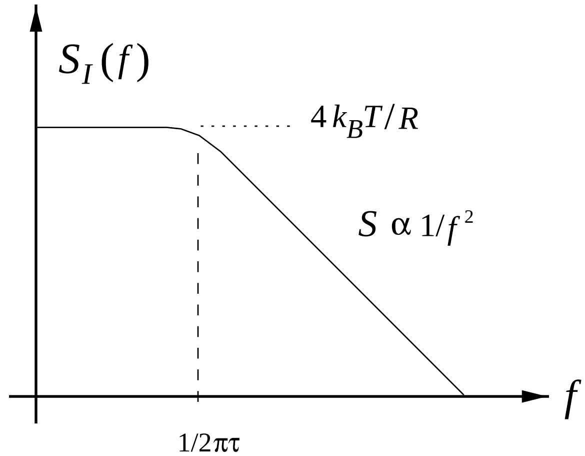

Figure 1: The frequency spectrum of the autocorrelation function for Thermal noise

at 1.6 THz which is well beyond the frequency that any integrated circuit can operate and therefore the spectrum can be considered *white* for all practical purposes. Now it is instructive to present Nyquist's original derivation of his theorem.

1.4 Nyquist's Development of Thermal Noise

The problem of Thermal noise in the resistor R can be viewed as a simple one-dimensional case of black-body radiation. One need only consider an ideal (lossless) one-dimensional transmission line of great length L which is terminated on both ends by resistances R with the whole system being in equilibrium at the temperature T. The transmission line is chosen so that its characteristic impedance is equal to R. Then any voltage wave propagating along the line is completely absorbed by the terminating resistor R without reflection. The resistor is then indeed the analog of a black body in one dimension. A voltage wave of the form $V = V_o \exp[i(\kappa x - \omega t)]$ propagates along the transmission line with a velocity $c' = \omega/\kappa$. To count the possible modes, we will consider the domain between x=0 and $x = L$ and impose the boundary condition $V(L) = V(0)$ on the possible propagating waves. Then $\kappa L = 2\pi n$, where n is any integer, and there are $\Delta n = (1/2\pi)d\kappa$ such modes *per unit length* of the line in the frequency range between ω and $\omega + d\omega$. The mean energy in each mode is given by the Plank distribution for bosons

$$E(\omega) = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \,, \tag{26}$$

where $\beta \equiv 1/k_BT$ is the standard symbol used in statistical mechanics textbooks.

Now we invoke the detailed balance argument by equating (in any small frequency range ω and $\omega + d\omega$) the power absorbed by a resistor to the power emitted by it. Since there are

$1/2\pi(d\omega/c')$ propagating modes per unit length in this frequency range, the mean energy per unit time incident upon a resistor in this frequency range is

$$\mathcal{P}_i = c' \left(\frac{1}{2\pi} \frac{d\omega}{c'} \right) E(\omega) = \frac{1}{2\pi} E(\omega)\, d\omega \; . \tag{27}$$

This is the power *absorbed* by the resistor. By the principle of detailed balance this must be equal to the power *emitted* by the resistor in this frequency range. But if the thermal emf generated by a resistor is V, this voltage induces a current $I = V/2R$ in the line. Hence the mean power emitted down the line and absorbed by the resistor at the other end is

$$R \left\langle I^2 \right\rangle = R \left\langle \frac{V^2}{4R^2} \right\rangle = \frac{1}{4R} \int_0^\infty S(\omega)\, d\omega \; , \tag{28}$$

or $S(\omega)d\omega/4R$ in the frequency range between ω and $\omega + d\omega$. Equating this to equation (27) gives

$$\begin{aligned} \frac{1}{4R} S(\omega) d\omega &= \frac{1}{2\pi} E(\omega)\, d\omega \\ S(\omega) &= \frac{2R}{\pi} \frac{\hbar\omega}{\boldsymbol{e}^{\beta\hbar\omega} - 1} \; , \end{aligned} \tag{29}$$

which is quantum-mechanically correct. Substituting for $\omega = 2\pi f$ into (29) yields

$$S_V(f) = 4R \frac{hf}{\boldsymbol{e}^{hf/k_B T} - 1} \; , \tag{30}$$

where h is Plank's constant. For $hf \ll k_B T$, the Plank distribution (26) becomes the classical energy $k_B T$ and then there is no frequency dependence and the noise can be thought of as *white*. At room temperature the quantum-mechanical correction applies at 6.25 THz. Since our circuits work at frequencies orders of magnitude less than this number, we can use a simple equation for Thermal noise gained from substituting the asymptotic expression for the Plank distribution into (30)

$$S_V(0) = 4 k_B T R \; . \tag{31}$$

We can see quite a difference in the mathematical form of the frequency cutoff between equations (25) and (30). Also note that though equation (25) is a current spectrum and equation (30) is a voltage spectrum, we can also express thermal noise in terms of a power spectral density (i.e. $P = I^2 R = V^2/R$). I will now give a physical argument based on power developed by Prof. Frensley to explain this difference. If we study the power delivered to an impedance-matched load from a Norton source, the ac signal power delivered will be

$P = \frac{1}{2}\, R\langle I^2\rangle$. But, only half of the source current will flow through the matched load, so we get a factor of 1/4. Therefore the power spectral density will be

$$S_P(f) = \frac{RS_I(f)}{8} = \frac{k_BT}{2}\left(\frac{1}{1+\omega^2\tau^2}\right)\ . \tag{32}$$

The thermal noise analysis by Nyquist given above proceeds from a thermodynamic argument that there should be a power density of $\frac{1}{2}k_BT\,df$ in any frequency interval df. However, this creates a problem because the total power should be the integral of S_P over all frequencies, and if S_P is a constant, then this integral will diverge. Thus, there must be some upper frequency cutoff to the noise spectrum. This is precisely the same problem as the "ultraviolet divergence" in the classical theory of black-body radiation. This divergence was removed by Plank with the introduction of the quantum hypothesis. (In fact, black-body radiation is simply thermal noise in a radiation field, and "black" means that the body is exactly impedance-matched to the electromagnetic radiation field.) Nyquist invoked the quantum limit, to find a frequency cutoff on the order of $k_BT/\hbar$.

The analysis of section 1.3, however, shows a cutoff that is dependent on the material properties of the resistor, through the relaxation time τ. This would appear to be in conflict with the more general treatment by Nyquist. There really is no contradiction, if we use the RLC equivalent circuit model of a resistor. When we associate the noise current generator with the ideal resistance, then the inductance due to electron inertia will obviously have some effect on the noise current observed at the terminals of the physical resistor. It is easy to show using circuit analysis with the time constant definition, $\tau = L/R$, that the spectral density in equation (25) will appear at the resistor terminals if a noise source having $S_I(f) = 4\,k_BT/R$ is placed across the ideal resistor.

In summary, there has to be a cutoff to the thermal noise spectrum in any physical device. The cutoff frequency will depend upon the details of electron transport in the device in question. However, these transport details will also reflect in the high-frequency impedance of that device, so that the Nyquist theorem in the form $S_P(f) = \frac{1}{2}\,k_BT$, is satisfied up to the quantum frequency limit.