MANUSCRIPT BOOK 1

OF

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(via Compositions [illegible])

~~$e^{-2\pi} = \frac{1}{540}\left[1+\left\{\frac{1}{126}\right.\right.$~~

$\frac{1}{540}\left[1+\frac{1}{120}\left\{1+\frac{1}{100}\left(1+\frac{1}{35}\cdot\overline{1+\frac{1}{90}+\frac{1}{90\cdot 25}}\right)\right\}\right]$

$$x\cot x = 1 - B_2\frac{2x^2}{\lfloor 2} - B_4\frac{2x^4}{\lfloor 4} - B_6\frac{2x^6}{\lfloor 6} - \&c$$

(a) $$-x\left\{\cot\frac{x}{2} + \cot\frac{x\omega}{2} + \cot\frac{x\omega^2}{2}\right\}$$

$$= 6\left\{\frac{B_4}{\lfloor 4}x^4 + \frac{B_{10}}{\lfloor 10}x^{10} + \frac{B_{16}}{\lfloor 16}x^{16} + \&c\right\}$$

$$= x\cdot\frac{\frac{x^6}{\lfloor 6} - \frac{x^{12}}{\lfloor 12} + }{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(b) $$-\frac{x}{2}\left\{\cot\frac{x}{2} + \omega^2\cot\frac{x\omega}{2} + \omega\cot\frac{x\omega^2}{2}\right\}$$

$$= 3\left\{\frac{B_2}{\lfloor 2}x^2 + \frac{B_8}{\lfloor 8}x^8 + \&c\right\}$$

$$= x\cdot\frac{\frac{x^4}{\lfloor 4} - \frac{x^{10}}{\lfloor 10} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(c) $$-\frac{x}{2}\left\{\cot\frac{x}{2} + \omega\cot\frac{x\omega}{2} + \omega^2\cot\frac{x\omega^2}{2}\right\}$$

$$= 3\left\{B_0 + \frac{B_6}{\lfloor 6}x^6 + \frac{B_{12}}{\lfloor 12}x^{12} + \&c\right\}$$

$$= -x\cdot\frac{\frac{x^2}{\lfloor 2} - \frac{x^8}{\lfloor 8} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

$$B_n - \frac{n}{4}\cdot\frac{n-1}{5}\cdots\frac{n-5}{9}B_{n-6} + \frac{n}{4}\cdot\frac{n-1}{5}\cdots\frac{n-11}{15}B_{n-12}$$

$$+ \&c = \frac{2}{(n+1)(n+2)}(-1)^{\frac{n-2}{6}} \text{ or}$$

$$\frac{2}{(n+1)(n+2)}(-1)^{\frac{n-6}{6}} \text{ or}$$

$$\frac{1}{(n+1)(n+2)}(-1)^{\frac{n-4}{6}}$$

$B_2 = \frac{1}{6}$

$B_8 - \frac{B_2}{3} = -\frac{1}{45}$

$B_{14} - \frac{143}{4} B_8 + \frac{B_2}{5} = \frac{1}{120}$

$B_4 = \frac{1}{30}$

$B_{10} - \frac{5}{2} B_4 = -\frac{1}{132}$

$B_{16} - \frac{286}{3} B_{10} + 4 B_4 = \frac{1}{306}$

$B_0 = -1$

$B_6 - \frac{B_0}{84} = \frac{1}{28}$

$B_{12} - 11 B_6 + \frac{B_0}{455} = -\frac{1}{91}$

$B_{18} - 221 B_{12} + \frac{204}{5} B_6 - \frac{B_0}{1330} = \frac{1}{190}$

$\frac{n+2}{3} B_n = {}^nC_6 B_{n-6} B_6 + {}^nC_{12} B_{n-12} B_{12}$ &c

when $n-2$ is a multiple of 6.

i-e

i $B_n - \frac{n(n-1)}{2.3}B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.3.4.5}B_{n-4} - \&c$

ii $B_n - \frac{n(n-1)}{3.4}B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{3.4.5.6}B_{n-4} + \frac{(-1)^{\frac{n}{2}}}{2} = 0$

iii $B_n - \frac{n(n-1)}{4.6}B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{4.6.8.10}B_{n-4} - \&c + \frac{(-1)^{\frac{n}{2}}}{n+1} = 0$

N.B. 0 is excluded in i.

$+ \frac{(-1)^{\frac{n}{2}}}{2^n} = 0$

~~$B_n = B_{n-2}\left\{\frac{n(n-1)}{40} + 33\,n(n-1) \times .0000100090\right\}$~~

~~$\log_e B_n = \log_e B_{n-2} + \log n + \log(n-1) - 2\log 2\pi$~~

$$B_n = \frac{n(n-1)}{4\pi^2}B_{n-2}\left(1-\frac{3}{2^n-1}\right)\left(1-\frac{8}{3^n-1}\right)\left(1-\frac{24}{5^n-1}\right)\left(1-\frac{48}{7^n-1}\right)$$

$$B_n = \frac{n(n-1)}{(n-2)(n-3)}\,\frac{B_{n-2}\cdot B_{n-2}}{B_{n-4}}\,\Pi\left\{1+\frac{(p^2-1)^2}{\left(p^n-1-p^4+\frac{p^4}{p^n}\right)}\right\}$$

$$(n+\tfrac{1}{2})\log_{10} n - 1.2324743503\,n + .700120 + \frac{.0362}{n}$$

$$-4\left\{B_2\frac{x^2}{\lfloor 2} + B_6\frac{x^6}{\lfloor 6} + B_{10}\frac{x^{10}}{\lfloor 10} + \&c\right\}$$

$$= x.\ \frac{\frac{x^3}{\lfloor 3} - \frac{x^7}{2^2\lfloor 7} + \frac{x^{11}}{2^4\lfloor 11} - \&c}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2\lfloor 6} + \frac{x^{10}}{2^4\lfloor 10} - \&c}$$

Sin

$\frac{x^7}{\lfloor 7}$

Sin

$$2\left\{B_0 + B_4\frac{x^4}{\lfloor 4} + B_8\frac{x^8}{\lfloor 8} + \&c\right\}$$

$$= -x.\ \frac{\frac{x}{\lfloor 1} - \frac{x^5}{2^2\lfloor 5} + \frac{x^9}{2^4\lfloor 9} - \&c}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2\lfloor 6} + \frac{x^{10}}{2^4\lfloor 10} - \&c}$$

$\sqrt{2} = 1.41421356237309504880 17$

~~$\sqrt{3} = 1.7320$~~

3 — $64x^{24}$ $\qquad$ 7 — $\frac{x^{24}}{64}$

3 — $x^3 = 4/2$

11 — $x^3 - x^2 + x = 4/2$

19 — $x^3 + x^2 = 4/2$

27 — $x^3 + x^2 \sqrt[5]{3} = 4/2$

35 —

43 — $x^3 + x = 4/2$

51 —

59 —

67 — $x^3 + x^2 + x = 4/2$

75 —

7 — $x^3 = 1$

15 —

23 — $x^3 + x^2 = 1$

31 — $x^3 + x = 1$

39 —

47

55

63

CHAPTER I.

MAGIC SQUARES

Let a be the average, s a row or a column, m middle [illegible] row or column or columns, d a diagonal and W [illegible] the whole sum.

I. When the square contains 3 rows and 3 columns,

i. If s and a are [illegible] write a in the middle and sup- ply the other figures.

Sol:- $d_1 + d_2 + m_1 + m_2 = W + 3x$ where x is the [illegible] of figure in middle.

$\therefore 4s = 3s + 3x \quad \therefore s = 3x \quad$ or $x = a$.

Cor. The figures in [illegible] are in A.P.

Sol:- The sum of the [illegible] d is s or 3a and 2nd = a $\therefore$ 1st + 3rd = 2a = twice the second [illegible] are in A.P. Similarly in m also.

Ex. 1. Fill up the square when s = 15

6	1	8
7	5	3
2	9	4

[illegible] 2. When s = 27 and all numbers are odd.

15	1	11
5	9	13
7	17	3

$d_1 + d_2 - s$ [illegible] middle

ii. When s and d are unequal write [illegible]

[illegible] the number [illegible] are in A.P. [illegible]

Ex. 1. [illegible]

Sol. [illegible]

2. Fill up the square when $S=20$, $d_1=14$ and $d_2=19$.

12	2	8
4	5	11
6	13	1

III. When the diagonals, columns and rows are all different

write $\frac{1}{3}(d_1+d_2+m_1+m_2-w)$ in the middle,

Sol. As in I 1 b.

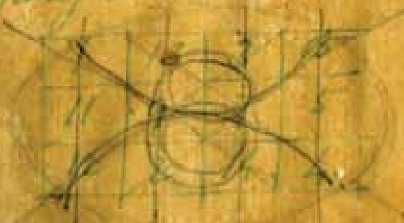
Ex. Fill up the square when d_1 [illegible] 5, $d_2=19$ and the columns and rows are 16, 17, 17, 6, 24, and 18.

4	2	3
8	9	4
7	6	5

7. When an oblong contains 3 rows and 4 columns.

	A+C = 2B+2D		
A	C+D	A+2D	C+2D
B+3D	B+D	B+2D	B
C	A+D	C+2D	A+3D

Ex: Fill up the oblong when $a=8$



3. When a square contains 4 rows and 4 columns.

i. When the diagonals, columns and rows are all different, arrange the central four so that the sum may be $\frac{1}{4}(d_1+d_2+m_1+m_2-w)$.

II. When [illegible]

Fig I

A+P	C+S	D+Q	B+R
D+R	B+Q	A+S	C+P
B+S	D+P	C+R	A+R
C+R	A+R	B+P	D+S

Fig II

A+P = B+Q C+S = A+R

A+P	D+R	D+R	A+S
B+S	C+R	C+R	B+P
C+S	B+R	B+R	C+P
D+P	A+R	A+R	B+S

Ex.

1	10	15	8
16	7	2	9
6	13	12	3
11	4	5	14

1	15	4	[illegible]
12	6	7	9
8	10	11	5
13	3	2	16

III. When a square contains 5 rows and 5 columns.

A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

1	58	59	4	5	62	63	8
[illegible]	55	54	13	12	51	50	9
24	47	46	21	20	43	42	17
25	34	35	28	29	38	39	32
33	26	27	36	37	30	31	40
[illegible]	23	22	45	44	19	18	41
52	[illegible]	[illegible]	[illegible]	[illegible]	11	10	49
57	2	3	60	61	6	7	64

Similarly [illegible] squares containing [illegible] number [illegible]

CHAPTER II

1. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots\cdots + \frac{1}{2n}$

$= [illegible] + \frac{1}{2^3 - 2} + \frac{1}{4^3 - 4} + \frac{1}{6^3 - 6} + \cdots\cdots + \frac{1}{(2n)^3 - 2n}$

Sol:- $\frac{1}{(2n)^3 - 2n} = \frac{1}{2}\cdot\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2(2n+1)}$

$\therefore$ R.H.S $= \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\cdots + \frac{1}{2n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\cdots + \frac{1}{2n}\right)$

$+ \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{5} \cdots\cdots + \frac{1}{2n+1}\right) + \frac{n}{2n+1} - \frac{1}{2}$

$= \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\cdots + \frac{1}{2n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\cdots + \frac{1}{2n}\right)$

$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} \cdots\cdots \frac{1}{n}\right)$

$= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots\cdots + \frac{1}{2n}$

Cor. $2\log_e 2 = 1 + \frac{2}{2^3 - 2} + \frac{2}{4^3 - 4} + \frac{2}{6^3 - 6} + \frac{2}{8^3 - 8} + \cdots\cdots$ ad inf.

Sol. R.H.S $= 2\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots\cdots + \frac{1}{2n}\right)$ when $n = \infty$

Let $n = \frac{1}{dx}$

then the given series $= \frac{2dx}{1+dx} + \frac{2dx}{1+2dx} + \cdots\cdots + \frac{2dx}{1+1}$

$= 2\int_1^2 \frac{1}{x}\, dx = 2\log_e 2.$

or thus —

In the Solution of II.1 we get $\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots \frac{1}{2n-1}\right)$

$- \left(\frac{1}{2} + \frac{1}{4} + \cdots\cdots \frac{1}{2n}\right)$. When $n = \infty$ this becomes

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c = \log_e 2$ $\therefore$ The reqd. Sum $= 2\log_e 2$

N.B. $\Sigma\frac{1}{n}$ means the sum of the reciprocals of n natural numbers. Therefore $\Sigma\frac{1}{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{2n}$ and $\neq$ $\frac{1}{2} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 3} + \cdots\cdots + \frac{1}{2n}$. $\Sigma\frac{1}{n}/2$ should not be written as $\Sigma\frac{n}{2n}$ which has no meaning according to our convention.

Ex. Show that $1 + \frac{n-1}{n+1} + \frac{n-2}{n+2} + \frac{n-3}{n+3} + \cdots\cdots + \frac{n-n}{n+n}$

$= 2n\left\{\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \cdots\cdots + \frac{1}{(2n-1)(2n)(2n+1)}\right\} - \frac{n}{2n+1}$

Sol. We have by II 1,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \frac{n}{2n+1} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{(2n-1)(2n)(2n+1)}$$

Multiplying both sides by $2n$

$$\frac{2n}{n+1} + \frac{2n}{n+2} + \cdots + \frac{2n}{2n} = \frac{2n^2}{2n+1} + 2n\left\{\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \text{\&c to } n \text{ terms}\right\}$$

$$\therefore \left(\frac{2n}{n+1} - 1\right) + \left(\frac{2n}{n+2} - 1\right) + \left(\frac{2n}{n+3} - 1\right) + \cdots + \left(\frac{2n}{2n} - 1\right)$$

$$= \frac{2n^2}{2n+1} - n + 2n\left\{\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \text{\&c to } n \text{ terms}\right\}$$

$$\therefore \frac{n-1}{n+1} + \frac{n-2}{n+2} + \frac{n-3}{n+3} + \cdots + \frac{n-n}{n+n}$$

$$= 2n\left\{\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \frac{1}{(2n-1)2n(2n+1)}\right\} - \frac{n}{2n+1}$$

2. $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} = 1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \cdots + \frac{2}{(3n)^3-3n}$

Sol. By proceeding as in II 1 we have R.H.S. $= \frac{1}{3n+1} - \frac{1}{n} = $ L.H.S.

i.e. $1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \frac{2}{9^3-9} + \text{\&c} = \log_e 3$.

3. $\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \cdots + \tan^{-1}\frac{1}{3n+1}$

$$= \tan^{-1} 1 + \tan^{-1}\frac{10}{5\cdot 8} + \tan^{-1}\frac{20}{76\cdot 35} + \cdots + \tan^{-1}\frac{10n}{(3n^2+2)(9n\ldots}$$

Cor. $\log_e 3 = \tan^{-1} 1 + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{69} + \tan^{-1}\frac{3}{282} + \tan^{-1}\frac{4}{715} + \text{\&c}$

4. $\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n+1}\right)$

$$= 1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \cdots + \frac{2}{(4n)^3-4n}$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4n+1}\right) + \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}\right)$$

Sol. By proceeding as in II 1 we have RHS $= \Sigma\frac{1}{4n+1} - \frac{1}{2}\Sigma\frac{1}{2n}$

$$- \frac{1}{2}\Sigma\frac{1}{n} = \Sigma\frac{1}{4n+1} - \Sigma\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$$

$$= \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{4n+1}\right) - \left(\frac{1}{2n+2} + \frac{1}{2n+4} + \cdots + \frac{1}{4n}\right)$$

$$= \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n+1}\right)$$

Again, $\Sigma\frac{1}{4n+1} - \frac{1}{2}\Sigma\frac{1}{n} - \frac{1}{2}\Sigma\frac{1}{2n} = \Sigma\frac{1}{4n+1} - 2\Sigma\frac{1}{2n} + \frac{1}{2}\Sigma\frac{1}{2n}$

$= \left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{4n+1}\right) - 2\left(\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{4n}\right)$

$+ \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n}\right) - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots+\frac{1}{2n}\right)$

$= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{4n+1}$

$+ \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots-\frac{1}{2n}\right).$

Cor. $1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\&c = \frac{3}{2}\log_e 2.$

5. $\frac{2}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right) + \left(\frac{1}{2n+1}+\frac{1}{2n+3}+\dots+\frac{1}{6n+1}\right)$

$= 1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\frac{2}{18^3-18}+\dots+\frac{2}{(6n)^3-6n}$

Sol. Proceeding as in Ex 1 the sum is $\Sigma\frac{1}{6n+1} - \frac{1}{2}\Sigma\frac{1}{3n}$

$-\frac{1}{3}\Sigma\frac{1}{2n} - \frac{1}{6}\Sigma\frac{1}{n} = $ L.H.S.

Cor. $1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\frac{2}{18^3-18}+\&c = \frac{1}{2}\log_e 3+\frac{1}{3}\log_e 4$

Ex. 1. $\frac{1}{4}\log_e 2 = \frac{1}{2^3-2}+\frac{1}{6^3-6}+\frac{1}{10^3-10}+\frac{1}{14^3-14}+\&c$

2. $\log_e 2 = 1-\frac{2}{2^3-2}+\frac{2}{4^3-4}-\frac{2}{6^3-6}+\&c$

3. $2\left\{1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\dots+\frac{2}{(4n)^3-4n}\right\}$

$= \left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\dots+\frac{2}{(4n)^3-4n}\right\} + \frac{1}{(4n+1)(4n+2)}$

$+ \frac{1}{2}\left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\dots+\frac{2}{(2n)^3-2n}\right\}$

4. Shew that $1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\dots+\frac{2}{(4n)^3-4n}$

$= \frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots\frac{1}{2n}\right) + \left(\frac{1}{2n+1}+\frac{1}{2n+2}+\dots\frac{1}{4n+1}\right)$

5. $\frac{1}{3^3-3}+\frac{1}{9^3-9}+\frac{1}{15^3-15}+\&c = \frac{1}{4}\log_e 3-\frac{1}{3}\log_e 2.$

6. $\frac{4}{3}\log_e 2 = 1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\&c$

7. Show that $2\left\{1 + \frac{1}{6^3-6} + \frac{1}{12^3-12} + \cdots + \frac{1}{(6n)^3-6n}\right\}$

$+ \frac{1}{3}\left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \cdots + \frac{2}{(2n)^3-2n}\right\}$

$= 1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \cdots + \frac{2}{(3n)^3-3n} + \frac{2}{(6n+1)(6n+2)(6n+3)}$

$+ 1 + \frac{2}{2^3-2} + \frac{2}{4^3-6} + \cdots + \frac{2}{(6n)^3-6n}$

8. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{7}$

$+ \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{1}{10} + \tan^{-1}\frac{1}{11} + \tan^{-1}\frac{1}{12} + \tan^{-1}\frac{1}{13}$

$= \frac{\pi}{2} + 2\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{49} + \tan^{-1}\frac{3}{232} + \tan^{-1}\frac{4}{715}$

9. $2\left(\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \cdots \tan^{-1}\frac{1}{2n+1}\right) = \tan^{-1}\frac{n+1}{n}$

$+ \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{4}{137} + \tan^{-1}\frac{6}{667} + \tan^{-1}\frac{8}{2081} +$

$\cdots + \tan^{-1}\frac{2n}{8n^4+2n^2+1} +$

$2\left(\tan^{-1}\frac{1}{1.7} + \tan^{-1}\frac{1}{2.19} + \tan^{-1}\frac{1}{3.39} + \cdots + \tan^{-1}\frac{1}{n(4n^2+3)}\right)$

10. $\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \cdots + \tan^{-1}\frac{1}{2n+1} + \tan^{-1}\frac{1}{2n+3}$

$+ \tan^{-1}\frac{1}{2n+5} + \cdots + \tan^{-1}\frac{1}{4n+1}$

$= \frac{\pi}{4} + \tan^{-1}\frac{9}{53} + \tan^{-1}\frac{18}{599} + \tan^{-1}\frac{27}{2789} + \cdots + \tan^{-1}\frac{9n}{32n^4+2n^2-1}$

$+ \tan^{-1}\frac{4}{137} + \tan^{-1}\frac{8}{2081} + \tan^{-1}\frac{12}{10441} + \cdots \tan^{-1}\frac{4n}{128n^4+8n^2+1}$

N.B. $1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \frac{2}{(3x)^3-3x} + \cdots + \frac{2}{(nx)^3-nx}$ cannot be expressed as in II 2. for all values of x but 2, 3, 4 and 6 though it can be summed up for all values of x when $n = \infty$. Refer to Chapter

6. If $H_n = S^n(n+\frac{1}{2}) - \frac{1}{2}$ then

$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{H_n} =$

$$n\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\cdots+\frac{2}{(3n)^3-3n}\right\}$$

$$+(n-1)\left\{\frac{2}{(3K_0+3)^3-(3K_0+3)}+\frac{2}{(3K_0+6)^3-(3K_0+6)}+\cdots+\frac{2}{(3K_1)^3-3K_1}\right\}$$

$$+(n-2)\left\{\frac{2}{(3K_1+3)^3-(3K_1+3)}+\frac{2}{(3K_1+6)^3-(3K_1+6)}+\cdots+\frac{2}{(3K_2)^3-3K_2}\right\}$$

$+\cdots\cdots$ to n terms

Sol. By II 2 we have,

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots\cdots+\frac{2}{(3n)^3-3n}$$

$$\frac{1}{3n+2}+\frac{1}{3n+3}+\cdots\cdots+\frac{1}{9n+4}=1+\frac{2}{3^3-3}+\cdots\cdots+\frac{2}{(9n+3)^3-(9n+3)}$$

$$\frac{1}{9n+5}+\frac{1}{9n+6}+\cdots\cdots+\frac{1}{27n+13}=1+\frac{2}{3^3-3}+\cdots\cdots+\frac{2}{(27n+12)^3-(27n+12)}$$

writing thus n times and then adding up all the terms we can get the result.

Cor. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{\frac{3^n-1}{2}}$

$$=n+(n-1)\left(\frac{2}{3^3-3}\right)+(n-2)\left(\frac{2}{6^3-6}+\frac{2}{9^3-9}+\frac{2}{12^3-12}\right)$$

$$+(n-3)\left(\frac{2}{15^3-15}+\frac{2}{18^3-18}+\cdots+\frac{2}{39^3-39}\right)+\text{ \&c to } n \text{ terms.}$$

N.B. II 1 & II 2 &c. are very useful in finding the approximate value of $\Sigma\frac{1}{x}$ whether n is small or very great without knowing logarithms, differential and integral calculus. In finding $\Sigma\frac{1}{x}$ it must be remembered that when a_1 and a_n are very great and a_1, a_2, a_3 &c are in A.P. the approximate value of $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$

$$=\frac{2n}{a_1+a_n}.$$

$$\text{Ex. 1}\quad 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}$$

$$=3+\frac{1}{5}+\frac{1}{105}+\frac{1}{360}+\frac{1}{858}$$

9. Show that $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1000}$
$= 7\frac{1}{2}$ very nearly.

7. $\tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+3)^2} + \tan^{-1}\frac{2}{(x+5)^2} +$ &c to n terms

$= \tan^{-1}\frac{2n}{x^2+2nx+1}$

Sol. $\tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2} = \tan^{-1}\frac{2}{(x+1)^2}$

$\therefore$ L.H.S $= \left(\tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2}\right) + \left(\tan^{-1}\frac{1}{x+2} - \tan^{-1}\frac{1}{x+4}\right)$

$+ \left(\tan^{-1}\frac{1}{x+4} - \tan^{-1}\frac{1}{x+6}\right) + \dots + \left(\tan^{-1}\frac{1}{x+2n-2} - \tan^{-1}\frac{1}{x+2n}\right)$

$= \tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2n} = \tan^{-1}\frac{2n}{x^2+2nx+1}$

Cor. $\tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+3)^2} + \tan^{-1}\frac{2}{(x+5)^2} +$ &c $= \tan^{-1}\frac{1}{x}$.

Sol. Make n infinite in Ex 7.

Ex. 1. $\tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+2)^2} + \tan^{-1}\frac{2}{(x+3)^2} +$ &c $= \tan^{-1}\frac{2x+1}{x^2+x-1}$

Sol. $\tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+3)^2} +$ &c $= \tan^{-1}\frac{1}{x}$

$\tan^{-1}\frac{2}{(x+2)^2} + \tan^{-1}\frac{2}{(x+4)^2} +$ &c $= \tan^{-1}\frac{1}{x+1}$

$\therefore \tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+2)^2} +$ &c $= \tan^{-1}\frac{2x+1}{x^2+x-1}$

N.B. If $x < \frac{\sqrt{5}-1}{2}$ add π to R.H.S.

2. $\tan^{-1}\frac{2}{(x+1)^2} - \tan^{-1}\frac{2}{(x+2)^2} + \tan^{-1}\frac{2}{(x+3)^2} -$ &c $= \tan^{-1}\frac{1}{x^2+x+1}$

3. $\tan^{-1}\frac{1}{2(x+1)^2} + \tan^{-1}\frac{1}{2(x+2)^2} + \tan^{-1}\frac{1}{2(x+3)^2} +$ &c $= \tan^{-1}\frac{1}{2x+1}$

4. $\frac{3\pi}{4} = \tan^{-1}\frac{2}{1^2} + \tan^{-1}\frac{2}{2^2} + \tan^{-1}\frac{2}{3^2} +$ &c

5. $\frac{\pi}{4} = \tan^{-1}\frac{1}{2\cdot1^2} + \tan^{-1}\frac{1}{2\cdot2^2} +$ &c $= \tan^{-1}\frac{2}{1^2} - \tan^{-1}\frac{2}{2^2} +$ &c

6. $\frac{\pi}{8} = \tan^{-1}\frac{1}{(1+\sqrt{2})^2} + \tan^{-1}\frac{1}{(1+2\sqrt{2})^2} + \tan^{-1}\frac{1}{(1+3\sqrt{2})^2} +$ &c

8. If $1 = A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \&c$

and $\begin{cases} P_n = A_1 P_{n-1} + A_2 P_{n-2} + A_3 P_{n-3} + \&c \cdots + A_{n-1} P_1 \\ Q_n = A_1 Q_{n-1} + A_2 Q_{n-2} + A_3 Q_{n-3} + \cdots + A_{n-1} Q_1 + A_n Q_0 \\ P_1 = 1 \text{ and } Q_0 = 1 \end{cases}$

then $\frac{P_n}{Q_n}$ approaches x when n becomes greater & greater.

Eg. 1. $x + x^2 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \Big| \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$, &c &c

2. $x + x^2 + x^3 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \Big| \frac{2}{4}, \frac{4}{7}, \frac{7}{13}, \frac{13}{24}, \frac{24}{44}$, &c

3. $x + x^3 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{1}, \Big| \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{6}, \frac{6}{9}, \frac{9}{13}, \frac{13}{19}$, &c

N.B. If $\frac{p}{q}$ and $\frac{r}{s}$ are two consecutive convergents to x then we may take $\frac{mp + nr}{mq + ns}$ in a suitable manner equivalent to x.

Ex. 1. Find convergents to $\log_e 2$.

Sol. Let $\log_e 2 = x$ then $e^x = 2$

$1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \&c$

$\therefore x = \frac{0}{1}, \frac{1}{1}, \Big| \frac{1}{1\frac{1}{2}}, \frac{1\frac{1}{2}}{2\frac{1}{6}}, \frac{2\frac{1}{6}}{3\frac{1}{8}}, \frac{3\frac{1}{8}}{4\frac{61}{120}}$ &c

$= \frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{9}{13}, \frac{52}{75}, \frac{375}{541}$ &c

2. When [illegible] $= x$ shew that the convergents to x are

$\frac{1}{2}, \frac{2}{7}, \frac{4}{7}, \frac{128}{61}$, &c

If $\phi(x) = e^x \psi(x)$, then

$$\phi(0) f(0) + \frac{\phi'(0) f'(0)}{\lfloor 1} + \frac{\phi''(0) f''(0)}{\lfloor 2} + \&c$$

$$\doteq \psi(0) f(1) + \frac{\psi'(0) f'(1)}{\lfloor 1} + \frac{\psi''(0) f''(1)}{\lfloor 2} + \&c$$

$$\frac{f(0)}{n} + \frac{f'(0)}{(n+1)\lfloor 1} + \frac{f''(0)}{(n+2)\lfloor 2} + \&c$$

$$= \frac{f(1)}{n} - \frac{f'(1)}{n(n+1)} + \frac{f''(1)}{n(n+1)(n+2)} - \&c$$

$$e^x \left\{ \frac{x}{1\lfloor 1} - \frac{2x^2}{2\lfloor 2} + \frac{2^2x^3}{3\lfloor 3} - \frac{2^3x^4}{4\lfloor 4} + \&c \right\}$$

$$= \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}\left(1+\frac{1}{3}\right) + \frac{x^4}{\lfloor 4}\left(1+\frac{1}{3}\right) + \frac{x^5}{\lfloor 5}\left(1+\frac{1}{3}+\frac{1}{5}\right)$$

$$+ \frac{x^6}{\lfloor 6}\left(1+\frac{1}{3}+\frac{1}{5}\right) + \&c$$

CHAPTER III

Let $\frac{1^n}{\underline{|1}}x + \frac{2^n}{\underline{|2}}x^2 + \frac{3^n}{\underline{|3}}x^3 + \frac{4^n}{\underline{|4}}x^4 + \&c = e^x f_n(x)$.

1. $\frac{x}{n\underline{|1}} + \frac{x^2}{(n+1)\underline{|2}} + \frac{x^3}{(n+2)\underline{|3}} + \frac{x^4}{(n+3)\underline{|4}} + \frac{x^5}{(n+4)\underline{|5}} + \&c$

$= e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$

Sol. $L.H.S = \frac{1}{x^{n-1}}\left\{\frac{x^n}{n\underline{|1}} + \frac{x^{n+1}}{(n+1)\underline{|2}} + \frac{x^{n+2}}{(n+2)\underline{|3}} + \&c\right\}$

$= \frac{1}{x^{n-1}}\int x^{n-1}e^x\,dx$

$= \frac{e^x}{x^{n-1}}\left\{\int x^{n-1}dx - \iint x^{n-1}(dx)^2 + \iiint x^{n-1}(dx)^3 - \&c\right\}$

$= \frac{e^x}{x^{n-1}}\left\{\frac{x^n}{n} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+2}}{n(n+1)(n+2)} - \&c\right\} = R.H.S$

Or thus:—

$\frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+m)} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+m)}$

$= \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \frac{m(m-1)(m-2)}{n(n+1)(n+2)(n+m)}$

&c &c.

$\therefore \frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \&c$

$\therefore \frac{1}{(n+m)\underline{|m}} = \frac{1}{n\underline{|m}} - \frac{1}{n(n+1)\underline{|m-1}} + \frac{1}{n(n+1)(n+2)\underline{|m-2}} - \&c$

But $\frac{1}{(n+m)\underline{|m}}$ is the coeff^t of x^{m+1} in L.H.S and the other is that of x^{m+1} in R.H.S. $\therefore$ L.H.S = R.H.S.

Cor. $\frac{x}{\underline{|1}} + \left(1+\frac{1}{2}\right)\frac{x^2}{\underline{|2}} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^3}{\underline{|3}} + \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^4}{\underline{|4}} + \&c$

$= e^x\left\{\frac{x}{1\underline{|1}} - \frac{x^2}{2\underline{|2}} + \frac{x^3}{3\underline{|3}} - \frac{x^4}{4\underline{|4}} + \&c\right\}$

Sol. $\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c = e^{-x}\left\{\frac{x}{n\underline{|1}} + \frac{x^2}{(n+1)\underline{|2}} + \&c\right\}$

Differentiating both sides with regards to n and [illegible] $n = 1$ we can get the result.

$$\ldots - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c$$

$$= \frac{f(x)}{n} - \frac{f_1(x)}{n^2} + \frac{f_2(x)}{n^3} - \frac{f_3(x)}{n^4} + \&c$$

Sol. By Th 1 we have,

$$e^{x}\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

$$= \frac{x}{n\,\underline{|1}} + \frac{x^2}{(n+1)\underline{|2}} + \frac{x^3}{(n+2)\underline{|3}} + \&c$$

changing n to $n+1$ we have

$$e^{x}\left\{\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c\right\}$$

$$= \frac{1}{x}\left\{\frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \&c\right\}$$

$$- \frac{1}{n^2}\left\{\frac{1}{\underline{|1}}x + \frac{2}{\underline{|2}}x^2 + \frac{3}{\underline{|3}}x^3 + \frac{4}{\underline{|4}}x^4 + \&c\right\}$$

$$+ \frac{1}{n^3}\left\{\frac{1^2}{\underline{|1}}x + \frac{2^2}{\underline{|2}}x^2 + \frac{3^2}{\underline{|3}}x^3 + \frac{4^2}{\underline{|4}}x^4 + \&c\right\}$$

$$- \&c = \frac{e^x}{n}f(x) - \frac{e^x}{n^2}f_1(x) + \frac{e^x}{n^3}f_2(x) - \&c$$ by our supposition. $\therefore$ L.H.S = R.H.S.

N.B. An easier solution for Th 1 is as follows –

Let $\phi(n) = \frac{x}{n\,\underline{|1}} + \frac{x^2}{(n+1)\underline{|2}} + \frac{x^3}{(n+2)\underline{|3}} + \&c$

Then $n\,\phi(n) = x + \frac{n}{n+1}\frac{x^2}{\underline{|2}} + \frac{n}{n+2}\frac{x^3}{\underline{|3}} + \&c$

and $x\,\phi(n+1) = \frac{x^2}{n+1} + \frac{x^3}{(n+2)\underline{|2}} + \&c$

$\therefore n\,\phi(n) + x\,\phi(n+1) = x + x^2 + \frac{x^3}{\underline{|2}} + \frac{x^4}{\underline{|3}} + \&c = x e^x$

$\therefore \phi(n) = e^x\frac{x}{n} - \frac{x}{n}\phi(n+1) = e^x\frac{x}{n} - e^x\frac{x^2}{n(n+1)} + \frac{x^2}{n(n+1)}\phi(n+2)$

&c &c

$$\therefore \frac{x}{n\,\underline{|1}} + \frac{x^2}{(n+1)\underline{|2}} + \frac{x^3}{(n+2)\underline{|3}} + \frac{x^4}{(n+3)\underline{|4}} + \&c$$

$$= e^{x}\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

3. $e^{x(e^a-1)} = 1 + \frac{a}{\underline{|1}} f_0(x) + \frac{a^2}{\underline{|2}} f_1(x) + \frac{a^3}{\underline{|3}} f_2(x) + \&c$

$\therefore e^{xe^a} = 1 + xe^a + \frac{x^2}{\underline{|2}} e^{2a} + \frac{x^3}{\underline{|3}} e^{3a} + \&c.$

The coeff.^{t} of a^{n+1} is $\frac{1}{\underline{|n+1}} \left\{ \frac{1^n}{\underline{|1}} x + \frac{2^n}{\underline{|2}} x^2 + \frac{3^n}{\underline{|3}} x^3 + \&c \right\}$

$= \frac{e^x}{\underline{|n+1}} f_n(x)$

$\therefore e^{xe^a} = e^x \left\{ 1 + \frac{a}{\underline{|1}} f_0(x) + \frac{a^2}{\underline{|2}} f_1(x) + \frac{a^3}{\underline{|3}} f_2(x) + \&c \right\}$.

4. $f_n(x) = x \left\{ 1 + n f_0(x) + \frac{n(n-1)}{\underline{|2}} f_1(x) + \frac{n(n-1)(n-2)}{\underline{|3}} f_2(x) + \&c \right\}$

Sol. Differentiating both sides in III 3 with regards to a

we have $xe^a e^{x(e^a-1)} = f_0(x) + \frac{a}{\underline{|1}} f_1(x) + \frac{a^2}{\underline{|2}} f_2(x) + \&c$

But $xe^a e^{x(e^a-1)} = xe^a \left\{ 1 + \frac{a}{\underline{|1}} f_0(x) + \frac{a^2}{\underline{|2}} f_1(x) + \&c \right\}$

Equating the coeff.^{ts} of a^n we get the result.

N.B. The above result may be written thus —

$f_0(x), f_1(x), f_2(x), f_3(x), f_4(x)$

a_0	b_0	c_0	d_0
	a_1	b_1	c_1
		a_2	b_2
			a_3

These are successive diff.^{s} a_n being equal to $x f_n(x)$

5. If $f_n(x) = \phi_1(n) x + \phi_2(n) x^2 + \phi_3(n) x^3 + \dots + \phi_{n+1}(n) x^{n+1}$.

Then $\frac{\phi_1(n)}{\underline{|0}} + \frac{\phi_2(n)}{\underline{|1}} + \frac{\phi_3(n)}{\underline{|2}} + \&c = \frac{2^n}{\underline{|n+1}}$

Sol. $e^x f_n(x) = e^x \left\{ \phi_1(n) x + \phi_2(n) x^2 + \dots + \phi_{n+1}(n) x^{n+1} \right\}$

Hence we have $e^x f_n(x) = \frac{1^n}{\underline{|0}} x + \frac{2^n}{\underline{|1}} x^2 + \frac{3^n}{\underline{|2}} x^3 + \&c$

by equating the coeff.^{ts} of x^2 in both sides we can get the result.

$$\phi(0) + \frac{\phi(1)}{\lfloor 1} x + \frac{\phi(2)}{\lfloor 2} x^2 +$$

$$= e^x \left\{ \phi(x) + \frac{x}{2} \phi''(x) + \ldots \right\} \text{ nearly}$$

6. $\phi_{n+1}(m) \lfloor n = (n+1)^m - n \cdot n^m + \frac{n(n-1)}{\lfloor 2}(n-1)^m - \frac{n(n-1)(n-2)}{\lfloor 3}(n-2)^m$

$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}(n-3)^m - \&c$

Sol. $f_n(x) = \phi_1(n) x + \phi_2(n) x^2 + \phi_3(n) x^3 + \&c$

$= e^{-x}\left\{ \frac{1^m}{\lfloor 1} x + \frac{2^m}{\lfloor 2} x^2 + \frac{3^m}{\lfloor 3} x^3 + \&c \right\}$

By equating the coeff^ts of x^{n+1} we can get the result.

7. $\phi_n(m+1) = n\phi_n(m) + \phi_{n-1}(m)$

Sol. $\phi_n(m+1) = \frac{1}{\lfloor n-1}\left\{ n^{m+1} - (n-1)(n-1)^{m+1} + \frac{(n-1)(n-2)}{\lfloor 2}(n-2)^{m+1} - \&c \right\}$

$\therefore \phi_n(m+1) - \phi_{n-1}(m) = \frac{1}{\lfloor n-1}\left\{ n \cdot n^m - n(n-1)(n-1)^m + \frac{n(n-1)(n-2)}{\lfloor 2}(n-2)^m - \&c \right\}$

$= n\phi_n(m)$. $\therefore \phi_n(m+1) = n\phi_n(m) + \phi_{n-1}(m)$. or in words this: write under each term the product of the coeff^t and the index of x of that term together with the coeff^t of the preceding one.

$f_0(x) = x$

$f_1(x) = x + x^2$

$f_2(x) = x + 3x^2 + x^3$

$f_3(x) = x + 7x^2 + 6x^3 + x^4$

$f_4(x) = x + 15x^2 + 25x^3 + 10x^4 + x^5$

$f_5(x) = x + 31x^2 + 90x^3 + 65x^4 + 15x^5 + x^6$

$f_6(x) = x + 63x^2 + 301x^3 + 350x^4 + 140x^5 + 21x^6 + x^7$

Ex. If $\frac{a_1}{n+1} - \frac{a_2}{(n+1)(n+2)} + \frac{a_3}{(n+1)(n+2)(n+3)} - \&c$

$= \frac{F(0)}{n} - \frac{F(1)}{n^2} + \frac{F(2)}{n^3} - \frac{F(3)}{n^4} + \&c$

Show that $F(m) = \phi_1(m) a_1 + \phi_2(m) a_2 + \phi_3(m) a_3 + \&c$

Show that $\phi_n(m)$ is the coeff^t of $\frac{x^m}{\lfloor m}$ in $\frac{e^x}{\lfloor n}(e^x - 1)^n$.

Sol. From III 6 we have $\phi_{n+1}(x)\lfloor n$

$= (n+1)^{n} - \frac{n}{\lfloor 1} n^{n} + \frac{n(n-1)}{\lfloor 2}(n-1)^{n} - \&c$

$=$ the coeff^{t}. of $\frac{x^n}{\lfloor n}$ in $\left\{ e^{x(n+1)} - \frac{n}{\lfloor 1} e^{xn} + \frac{n(n-1)}{\lfloor 2} e^{x(n-1)} \&c \right\}$

$=$ the coeff^{t}. of $\frac{x^n}{\lfloor n}$ in $e^{x}(e^{x}-1)^{n}$.

3. $\frac{d f_{n+1}(x)}{dx} = n f_{n-2}(x) + \frac{n(n-1)}{\lfloor 2} f_{n-3}(x) + \frac{n(n-1)(n-2)}{\lfloor 3} f_{n-4}(x) + \&c$.

Sol. Differentiating both sides in III 3 with regards to x and then differentiating the result with regards to a and then equating the coeff^{ts}. as in III 4. we can get the result.

4. $\frac{1}{2} f_n(x) + \int f_n(x)\,dx = \frac{f_{n+1}(x)}{n+1} + \frac{B_2}{\lfloor 2} n f_{n-1}(x)$

$- \frac{B_4}{\lfloor 4} n(n-1)(n-2) f_{n-3}(x) + \frac{B_6}{\lfloor 6} n(n-1)(n-2)(n-3)(n-4) f_{n-5}(x)$

$- \&c$

Sol. Integrating both sides in III 3 with regards to x we have $\frac{1}{e^a-1}\left\{1 + \frac{a}{\lfloor 1} f_1(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c\right\}$

$= \frac{1}{e^a-1} + x + \frac{a}{\lfloor 1}\int f_0(x)\,dx + \frac{a^2}{\lfloor 2}\int f_1(x)\,dx + \frac{a^3}{\lfloor 3}\int f_2(x)\,dx + \&c$

Equating the coeff^{ts}. of a^{n+1} in both sides we can get the result.

5. If $\frac{1^n}{\lfloor 1} + \frac{2^n}{\lfloor 2} + \frac{3^n}{\lfloor 3} + \frac{4^n}{\lfloor 4} + \&c = e A_n$

Shew that $A_0 = 1$, $A_1 = 2$, $A_2 = 5$, $A_3 = 15$, $A_4 = 52$, $A_5 = 203$, $A_6 = 877$, $A_7 = 4140$, $A_8 = 21147$ &c

Sol. $2 = 1 + 1$, $5 = 1 + 2.1 + 2$, $15 = 1 + 3.1 + 3.2 + 5$, $52 = 1 + 4.1 + 6.2 + 4.5 + 15$, $203 = 1 + 5.1 + 10.2 + 10.5 + 5.15 + 52$ &c

6. If $-\frac{1^n}{\lfloor 1} + \frac{2^n}{\lfloor 2} - \frac{3^n}{\lfloor 3} + \frac{4^n}{\lfloor 4} - \&c = \frac{A_n}{e}$, shew that $A_0 = -1$, $A_1 = 0$, $A_2 = 1$, $A_3 = 1$, $A_4 = -2$, $A_5 = -9$, $A_6 = -9$, $A_7 = 50$

A_8 267 &c &c

N.B.

1	2	5	15	52	203	877
	1	3	10	37	151	674
		2	7	27	114	523
			5	20	87	409
				15	67	322
					52	255
						203

-1	0	1	1	-2	-9	-9	50
	1	1	0	-3	-7	0	59
		0	-1	-3	-4	7	59
			-1	-2	-1	11	52
				-1	1	12	41
					2	11	29
						9	18
							9

7. Shew that

i. $\frac{1^3}{\underline{|1}} + \frac{2^3}{\underline{|2}} + \frac{3^3}{\underline{|3}} + \frac{4^3}{\underline{|4}} + \&c = 3\left(\frac{1^2}{\underline{|1}} + \frac{2^2}{\underline{|2}} + \frac{3^2}{\underline{|3}} + \&c\right)$

ii. $\frac{1^4(1^3+1)}{\underline{|1}} + \frac{2^4(2^3+1)}{\underline{|2}} + \frac{3^4(3^3+1)}{\underline{|3}} + \&c = 4\left(\frac{1^6}{\underline{|1}} + \frac{2^6}{\underline{|2}} + \frac{3^6}{\underline{|3}} + \&c\right)$

iii. $\frac{1^3}{\underline{|1}} - \frac{2^3}{\underline{|2}} + \frac{3^3}{\underline{|3}} - \frac{4^3}{\underline{|4}} + \&c = \frac{1^2}{\underline{|1}} - \frac{2^2}{\underline{|2}} + \frac{3^2}{\underline{|3}} - \frac{4^2}{\underline{|4}} + \&c$

iv. $\frac{1^6}{\underline{|1}} - \frac{2^6}{\underline{|2}} + \frac{3^6}{\underline{|3}} - \frac{4^6}{\underline{|4}} + \&c = \frac{1^5}{\underline{|1}} - \frac{2^5}{\underline{|2}} + \frac{3^5}{\underline{|3}} - \frac{4^5}{\underline{|4}} + \&c$

v. $\frac{1^3(1^2+1)(1^3+1)}{\underline{|1}} - \frac{2^3(2^2+1)(2^3+1)}{\underline{|2}} + \frac{3^3(3^2+1)(3^3+1)}{\underline{|3}} - \&c$

$= 6\left(\frac{1^7}{\underline{|1}} - \frac{2^7}{\underline{|2}} + \frac{3^7}{\underline{|3}} - \frac{4^7}{\underline{|4}} + \frac{5^7}{\underline{|5}} - \&c\right)$

8. If $\frac{x}{x+a+b} - \frac{bx^2}{(x+a+b)(x+a+2b)} + \frac{b^2x^3}{(x+a+b)(x+a+2b)(x+a+3b)}$

$- \&c = \frac{F_0(x)}{x} - \frac{F_1(x)}{x^2} + \frac{F_2(x)}{x^3} - \frac{F_3(x)}{x^4} + \&c$, then

$(a+b)^n \frac{x}{\underline{|1}} + (a+2b)^n \frac{x^2}{\underline{|2}} + (a+3b)^n \frac{x^3}{\underline{|3}} + \&c = e^x F_n(x)$

Cor 1. $F_0(x) + y F_1(x) + \frac{y^2}{\underline{|2}} F_2(x) + \frac{y^3}{\underline{|3}} F_3(x) + \&c = x e^{y(a+b)} e^{x(e^{by}-1)}$

Cor 2. $F_{n+1}(x) - (a+b)F_n(x) = bx\left\{F_n(x) + \frac{n}{\underline{|1}} F_{n-1}(x) b + \frac{n(n-1)}{\underline{|2}} F_{n-2}(x) b^2 + \&c\right\}$

Cor 3. If $F_n(x) = \phi_1(n)x + \phi_2(n)x^2 + \phi_3(n)x^3 + \&c$, then

$\frac{\phi_n(n)}{\underline{|0}} - \frac{\phi_{n-1}(n)}{\underline{|1}} + \frac{\phi_{n-2}(n)}{\underline{|2}} + \&c = \frac{(a+nb)^n}{\underline{|n-1}}$.

N.B. If $F_{n+1}(x) - (a+b)F_n(x) = \psi_n(x)$, then

$\psi_0(x)$	$\psi_1(x)$	$\psi_2(x)$	$\psi_3(x)$	$\psi_4(x)$	$\psi_5(x)$
	a_1	b_1	c_1	d_1	e_1
		a_2	b_2	c_2	d_2
			a_3	b_3	c_3
				a_4	b_4
					a_5

These are successive diffces., b times the previous term being subtracted from each term and a_n being equal to $b \times F_n(x)$.

Cor. 4. $\phi_n(x)\lfloor n-1 = (a+nb)^n - \frac{n-1}{\lfloor 1}(a+\overline{n-1}\,b)^n + \frac{(n-1)(n-2)}{\lfloor 2}(a+\overline{n-2}\,b)^n - \frac{(n-1)(n-2)(n-3)}{\lfloor 3}(a+\overline{n-3}\,b)^n + \&c$

Cor. 5. $\phi_n(n+1) = (a+nb)\,\phi_n(n) + b\,\phi_{n-1}(n)$ or in words thus — Write under each term the product of $a+nb$, n being the index of x, and the coeffᵗ. of x of that term together with b times the coeffᵗ. of the preceding one.

$F_0(x) = x$

$F_1(x) = (a+b)x + bx^2$

$F_2(x) = (a+b)^2x + b(2a+3b)x^2 + b^2x^3$

$F_3(x) = (a+b)^3x + b\{3(a+b)(a+2b)+b^2\}x^2 + 3b^2(a+2b)x^3 + b^3x^4$

$F_4(x) = (a+b)^4x + b\{2(a+b)(a+2b)+b^2\}(2a+3b)x^2 + b^2\{6(a+2b)^2+b^2\}$
$\times x^3 + 2b^3(2a+5b)x^4 + b^4x^5.$

Ex. 1. Show that $\phi_{n+1}(x)$ is the coeffᵗ. of $\frac{x^n}{\lfloor n}$ in $\frac{e^{x(a+b)}}{\lfloor n}(e^{bx}-1)^n$.

2. Shew that i. $\frac{1^3+1^2}{\lfloor 0} - \frac{3^3+3^2}{\lfloor 1} + \frac{5^3+5^2}{\lfloor 2} - \&c = 0$

ii. $\frac{1^4}{\lfloor 0} + \frac{2^4}{\lfloor 1} + \frac{3^4}{\lfloor 2} + \frac{4^4}{\lfloor 3} + \&c = 4\left(\frac{1^2}{\lfloor 0} + \frac{3^2}{\lfloor 1} + \frac{5^2}{\lfloor 2} + \&c\right)$

iii. $\frac{1^7+1^6}{\lfloor 0} - \frac{2^7+2^6}{\lfloor 1} + \frac{3^7+3^6}{\lfloor 2} - \&c = \frac{1^6}{\lfloor 0} - \frac{3^6}{\lfloor 1} + \frac{5^6}{\lfloor 2} - \frac{7^6}{\lfloor 3} + \&c$

iv. $1^4 - \frac{3^4}{1\lfloor 1} + \frac{5^4}{2\lfloor 2} - \frac{7^4}{3\lfloor 3} + \&c = \left(1 - \frac{1}{1\lfloor 1} + \frac{1}{2\lfloor 2} - \frac{1}{3\lfloor 3} + \&c\right) - 4.$

CHAPTER IV

1. If $x^r + \frac{(x+1)^{r+1}}{a\,1!} + \frac{(x+2)^{r+2}}{a^2\,2!} + \frac{(x+3)^{r+3}}{a^3\,3!} + \&c = F_r(x)$, then

$$F_{r+1}(x) = x\,F_r(x) + \frac{1}{a}F_{r+1}(x+1).$$

Sol. $F_{r+1}(x) = x^{r+1} + \frac{(x+1)^{r+2}}{a\,1!} + \frac{(x+2)^{r+3}}{a^2\,2!} + \&c$

$$= x\left\{x^r + \frac{(x+1)^{r+1}}{a\,1!} + \frac{(x+2)^{r+2}}{a^2\,2!} + \&c\right\} + \frac{1}{a}\left\{(x+1)^{r+1} + \frac{(x+2)^{r+2}}{a\,1!} + \&c\right\}$$

$$= x\,F_r(x) + \frac{1}{a}F_{r+1}(x+1).$$

We see from this identity that if we are able to find the sum for one value of r we can sum up the series for all values of r.

2. If $x = a\log_e x$, then $\frac{x^n}{n} = n^{-1} + \frac{(n+1)^0}{a\,1!} + \frac{(n+2)^1}{a^2\,2!} + \frac{(n+3)^2}{a^3\,3!} + \&c$

Sol. Suppose $f(n) = 1 + \frac{n}{a\,1!} + \frac{n(n+2)}{a^2\,2!} + \frac{n(n+3)^2}{a^3\,3!} + \&c$

If we multiply $f(n)$ by $f(m)$ we get $f(m+n)$.

$\therefore f(n) = \{f(1)\}^n$. Let $f(1) = x$ then $x^n = f(n)$.

$\frac{f(n)-1}{n}$, when $n=0$, $= \frac{1}{a} + \frac{2}{a^2\,2!} + \frac{3^2}{a^3\,3!} + \&c$

$$= \frac{1}{a}\left(1 + \frac{1}{a\,1!} + \frac{3}{a^2\,2!} + \frac{4^2}{a^3\,3!} + \&c\right) = \frac{1}{a}f(1) = \frac{x}{a}$$

i.e. $\frac{x^n-1}{n}$ when $n=0$, $= \frac{x}{a}$ or $\log_e x = \frac{x}{a}$ or $x = a\log_e x$.

N.B. The minimum value of $\frac{x}{\log_e x}$ is e. If $a=e$ $f(n)=e^n$.

If $a > e$ it is convergent but if $a < e$ it is divergent.

Cor. $e^x = 1 + \frac{x}{e^n} + \frac{x(x+2n)}{e^{2n}\,2!} + \frac{x(x+3n)^2}{e^{3n}\,3!} + \frac{x(x+4n)^3}{e^{4n}\,4!} + \&c$

Sol. Write e^x for x and $\frac{x}{n}$ for n in IV 2.

N.B. In a similar manner we can prove that if $aqx^p - x^q + 1 = 0$, then

$$x^n = 1 + \frac{n}{1!}a + \frac{n(n+2p-q)}{2!}a^2 + \frac{n(n+3p-q)(n+3p-2q)}{3!}a^3 + \frac{n(n+4p-q)(n+4p-2q)(n+4p-3q)}{4!}a^4 + \&c.$$

Ex. 1. Shew that

$$2^m = 1 + \frac{m}{2^n} + \frac{m(m+2n-1)}{2^{2n}\,|\underline{2}} + \frac{m(m+3n-1)(m+3n-2)}{2^{3n}\,|\underline{3}} + \&c$$

2. $$\left(\frac{2}{1+\sqrt{1-4x}}\right)^n = 1 + nx + \frac{n(n+3)}{|\underline{2}}x^2 + \frac{n(n+4)(n+5)}{|\underline{3}}x^3 + \&c$$

3. Expand x in terms of a in each of the following

i. $x^a = e^{\pm x}$; Sol. $a \log_e x = \pm x$.

ii. $x^a = a^{\pm x}$; Sol. $a \log_e x = \pm x \log_e a$ $\therefore \frac{x}{\log_e x} = \pm \frac{a}{\log_e a}$

iii. $x = a e^{\pm x}$; Sol. Let $x = \log_e y$, then $\log_e y = a y^{\pm 1}$

iv. $x = a^{\pm x}$; Sol. Let $x \log_e a = \log_e y$ then $\log_e y = y^{\pm 1} \log_e a$.

v. $x^{\pm x} = a$; Sol. Let $x = \frac{1}{y}$ then $y = a^{\mp y}$

vi. $x e^{\pm x} = a$; Sol. Let $x = \log_e y$ then $\log_e y = a y^{\pm 1}$

vii. $e^x \pm x = a$. Sol. Let $x = \log_e \log_e y$ then $e^a = y(\log_e y)^{\pm 1}$

viii. $x \pm \log_e x = a$; Sol. Let $x = \log_e y$ then $e^a = y(\log_e y)^{\pm 1}$

4. Shew how to find the values of the following for special values of x

i. $x^{x^{x^{\&c}}}$; Sol. Let its value be equal to v then $x^v = v$.

ii. $x \pm e^{x \pm e^{x \pm \&c}}$
Sol. $x \pm e^v = v$

iii. $\log_e \{x \log_e [x \log_e (x \ldots\ldots \text{ad inf.})]\}$

iv. $\pm \log_e \{x \pm \log_e [x \pm \log_e (x \pm \&c)]\}$

5. If we write $x^n \phi_n(x)$ for $F_n(x)$, we see that

$$\phi_{n+1}(x) - \log_e x\, \phi(n+1, x+1) = x\, \phi(n, x).$$

[illegible]

Case I. If r is positive

Let $\phi_r(n) = \frac{\psi_1(r,n)}{(1-\log_e x)^{n+1}} + \frac{\psi_2(r,n)}{(1-\log_e x)^{n+2}} + \dots + \frac{\psi_{r+1}(r,n)}{(1-\log_e x)^{2r+1}}$

Then by IX.1 we see that

$$n\,\psi_r(r,n) + \psi_{r-1}(r+1,n+1) = \psi_r(r+1,n+1) + \psi_r(r+1,n)$$

Case II. If r is negative the terms in R.H.S continue so far as the term independent of $(1-\log_e x)$.

N.B. $F_r(x)$ is convergent when $a > e$ or $x \not> e$ according as r is positive or not.

Cor. 1. $\psi_1(r,n) + \psi_2(r,n) + \psi_3(r,n) + \&c$ as far as the terms cease to continue is $\phi_r(n) = n^r$.

Sol. L.H.S $= \phi_r(n)$ when $x = 1$ i.e $F_r(x)$ when $x = 1$ i.e $F_r(n)$ when $a = \infty$ i.e n^r.

Cor. 2. $e^x = (1-x)\left\{1 + \frac{x+n}{e^x \lfloor 1} + \frac{(x+2n)^2}{e^{2x}\lfloor 2} + \frac{(x+3n)^3}{e^{3x}\lfloor 3} + \&c\right\}$

N.B. $\phi(-1,n) = \frac{1}{n}$

$$\phi(-2,n) = \frac{1-\log_e x}{n(n+1)} + \frac{1}{n^2(n+1)}$$

$$\phi(-3,n) = \frac{(1-\log_e x)^2}{n(n+1)(n+2)} + \frac{(3n+2)(1-\log_e x)}{n^2(n+1)^2(n+2)} + \frac{8n+2}{n^3(n+1)^2(n+2)}$$

$$\phi(0,n) = \frac{1}{1-\log_e x}$$

$$\phi(1,n) = \frac{n-1}{(1-\log_e x)^2} + \frac{1}{(1-\log_e x)^3}$$

$$\phi(2,n) = \frac{(n-1)(n-2)}{(1-\log_e x)^3} + \frac{(n-1)(n-2)\left(\frac{1}{n-2} + \frac{2}{n-1}\right)}{(1-\log_e x)^4} + \frac{1.3}{(1-\log_e x)^5}$$

$$\phi(3,n) = \frac{(n-1)(n-2)(n-3)}{(1-\log_e x)^4} + \frac{(n-1)(n-2)(n-3)\left(\frac{1}{n-3} + \frac{2}{n-2} + \frac{3}{n-1}\right)}{(1-\log_e x)^5} + \frac{15n-25}{(1-\log_e x)^6} + \frac{1.3.5}{(1-\log_e x)^7}$$

To expand x^m in ascending powers of h when

$$x^x = a^a e^h$$

Let $\left(\frac{x}{a}\right)^{\frac{x}{a}} = \frac{A_1}{1}\frac{h}{a} - \frac{A_2}{2}\left(\frac{h}{a}\right)^2 + \frac{A_3}{3}\left(\frac{h}{a}\right)^3 - \&c$

and let $n = \frac{1}{1-\log_e a}$

then $A_n - n(n-1)A_{n-1} = n\left\{2A_1A_{n-1} + \frac{n(n-1)}{\underline{|2}}A_2A_{n-2} + \frac{n(n-1)(n-2)}{\underline{|3}}A_3A_{n-3} + \&c\right\}$ the last term being

$\frac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}A_{\frac{n-1}{2}}$ or $\frac{\underline{|n}}{2\left(\underline{|\frac{n}{2}}\right)^2}A_{\frac{n}{2}}$ according as n is odd or even.

$A_1 = n$

$A_2 = n^3$

$A_3 = 3n^5 + n^4$

$A_4 = 15n^7 + 10n^6 + 2n^5$

$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6$

$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7$

$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8$

Multiply the power and the coeff^t. write under it the sum of this product and $(n-1)$ times the coeff^t. of the preceding term where n is the suffix of A.

N.B. For $\frac{a}{x}$ take $n+1$ times the coeff^t.; for $\log_e\frac{x}{a}$ take n times the coeff^t. and generally for $\frac{x^m}{a^m}$ take $n-m$ times the coeff^t.

Ex 1. Show that the sum of the coeff^{ts} of A_n is $(n-1)^{n-1}$

Sol. Put 1 for a. Then $x^x = e^h$. Let $x = \frac{1}{y}$

Then $y^{-\frac{1}{y}} = e^{-h}$ or $\frac{1}{y}\log_e y = -h$

By applying IV 2 we have $\frac{1}{y} = x = 1 + h - \frac{1}{\underline{|2}}h^2 + \frac{2^2}{\underline{|3}}h^3 - \frac{3^3}{\underline{|4}}h^4 + \&c$ ∴ The sum of the coeff^{ts} of $A_n = (n-1)^{n-1}$.

2. Expand x in ascending powers of h when $\frac{x}{a} = e^{h\frac{x}{a}}$.

[illegible] then $y^y = e^{-h}\left(\frac{1}{a}\right)^a$.

CHAPTER V

Let $F_1(x) = e^x - 1$, $F_2(x) = e^{e^x-1} - 1$, $F_3(x) = e^{e^{e^x-1}-1} - 1$, $F_4(x) = e^{e^{e^{e^x-1}-1}-1} - 1$, $F_5(x) = e^{e^{e^{e^{e^x-1}-1}-1}-1} - 1$ &c &c

Now let us try to find the expansions of $F_n(x)$

(1) In ascending powers of x }
(2) In ascending powers of n }

1\. Let $F_n(x) = x\phi_1(n) + x^2\phi_2(n) + x^3\phi_3(n) + \&c$
$= f_0(x) + n f_1(x) + n^2 f_2(x) + n^3 f_3(x) + \&c$

then $\log_e\left[1 + \log_e\left\{1 + \log_e\left(1 + \cdots + \log_e\overline{1+x}\right)\right\}\right]$ logarithms being taken n times

$= F_{-n}(x) = x\phi_1(-n) + x^2\phi_2(-n) + x^3\phi_3(-n) + \&c$
$= f_0(x) - n f_1(x) + n^2 f_2(x) - n^3 f_3(x) + \&c$

Sol. We have $e^{F_{n-1}(x)} - 1 = F_n(x)$ $\therefore F_{n-1}(x) = \log_e\{1 + F_n(x)\}$

$\because F_0(x) = x$ $\therefore F_{-1}(x) = \log_e(1+x)$ $\therefore F_{-2}(x) = \log_e\{1 + \log_e(1+x)\}$ &c &c

Cor. $F_0(x) = x$ and $f_0(x) = x$.

2\. $\dfrac{dF_n(x)}{dx} \div \dfrac{dF_{n-1}(x)}{dx} = 1 + F_n(x)$

Sol. $\because e^{F_{n-1}(x)} - 1 = F_n(x)$ $\therefore F_{n-1}(x) = \log_e\{1 + F_n(x)\}$

Differentiating both sides with regards to x we have

$\dfrac{dF_n(x)}{dx} = \{1 + F_n(x)\}\dfrac{dF_{n-1}(x)}{dx}$

$\dfrac{dF_n(x)}{dx} = \{1 + F_1(x)\}\{1 + F_2(x)\}\{1 + F_3(x)\}\cdots\{1 + F_n(x)\}.$

Sol. From (V, 2 cor. ...) $F_n'(x) = \{1+F_1(x)\} F_{n-1}'(x)$ [illegible]

$= \{1+F_1(x)\}\{1+F_2(x)\} F_{n-2}'(x)$ [illegible]

$=$ &c &c [illegible]

But from V. 1 cor. $F_0(x) = x$ $\therefore F_0'(x) = 1$

Cor. 2. $n\{\phi_n(x) - \phi_n(n-1)\} = (n-1)\phi_1(x)\phi_{n-1}(n-1) + (n-2)\phi_2(x)\phi_{n-2}(n-1)$

$+ (n-3)\phi_3(x)\phi_{n-3}(n-1) +$ &c.

Sol. $F_{n-1}'(x)\{1+F_1(x)\} = F_n'(x)$. Hence equate the coeff^ts. of x^{n+1}

Cor. 3. $\dfrac{d f_1(x)}{dx} = x - \frac{1}{2} f_1(x) + \frac{B_2}{\underline{|2}} f_2(x) - \frac{B_4}{\underline{|4}} f_4(x) + \frac{B_6}{\underline{|6}} f_6(x) -$ &c

Sol. From V. 2 cor. we have $F_n'(x) = \{1+F_1(x)\}\{1+F_2(x)\}\cdots\{1+F_n(x)\}$

i.e. $1 + n\dfrac{d f_1(x)}{dx} +$ &c $= e^{F_0(x) + F_1(x) + \cdots + F_n(x)}$

$\therefore \log_e\left\{1 + n\dfrac{d f_1(x)}{dx} + \text{\&c}\right\} = F_0(x) + F_1(x) + F_2(x) +$ &c to n terms.

$= \psi(x) + \int_0^n F_n(x)\,dn - \frac{1}{2} F_n(x) + \frac{B_2}{\underline{|2}} \dfrac{d F_n(x)}{dn} - \frac{B_4}{\underline{|4}} \dfrac{d^3 F_n(x)}{dn^3} +$ &c

where $\psi(x)$ is a function of x independent of n.

Equating the coeff^ts. of n, we have

$\dfrac{d f_1(x)}{dx} = x - \frac{1}{2} f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) -$ &c.

N. B. $\psi'(x) = \displaystyle\int_0^x \frac{x - \frac{d f_1 x}{dx}}{f_1(x)}\, dx$

Sol. Since when $n = 0$, $\log_e\left\{1 + n\dfrac{d f_1 x}{dx} + \text{\&c}\right\} = 0$

$\psi(x) = \frac{x}{2} - \frac{B_2}{2} f_1(x) + \frac{B_4}{4} f_3(x) - \frac{B_6}{6} f_5(x) +$ &c

$\psi'(x) = \frac{1}{2} - \frac{B_2}{2} f_1'(x) + \frac{B_4}{4} f_3'(x) - \frac{B_6}{6} f_5'(x) +$ &c.

$$\left\{1+2\left(\frac{\cos\theta}{\cosh\pi}+\frac{\cos 2\theta}{\cosh 2\pi}+\frac{\cos 3\theta}{\cosh 3\pi}+\&c\right)\right\}^{-2}$$

$$+\left\{1+2\left(\frac{\cosh\theta}{\cosh\pi}+\frac{\cosh 2\theta}{\cosh 2\pi}+\frac{\cosh 3\theta}{\cosh 3\pi}+\&c\right)\right\}^{-2}$$

~~is constant~~ $= \frac{2}{\pi}\left(1-\frac{1}{4}\right)^4.$

35. $$(\sqrt{5}-2)^8\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{36}(6-\sqrt{35})^6$$

$$\times\left(\sqrt{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\pm\sqrt{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\right)^2$$

---

21. $$\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{24}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{12}(2-\sqrt{3})^4$$

$$\times\left(\frac{\sqrt{3+\sqrt{7}}-\sqrt[4]{6\sqrt{7}}}{\sqrt{3+\sqrt{7}}+\sqrt[4]{6\sqrt{7}}}\right)^{12}$$

$\therefore \psi(x) f_1(x) = \frac{1}{2} f_1(x) - \frac{B_2}{2} f_1(x) f_1'(x) + \frac{B_4}{4} f_1(x) f_1'''(x) - \&c$

$= \frac{1}{2} f_1(x) - B_2 f_2(x) + B_4 f_4(x) - \&c$ by V 2.

$= x f_1'(x)$ by V 2 cor 3...

$\therefore \psi'(x) = \frac{x f_1'(x)}{f_1(x)}$ $\therefore \psi(x) = \int_0^x \frac{x f_1'(x)}{f_1(x)} dx$

3. $\frac{dF_n(x)}{dn} = f_1(x)\frac{dF_n(x)}{dx}$ and consequently $n f_n(x) = f_1(x)\frac{df_{n-1}(x)}{dx}$

Sol. In V 1. write $F_k(x)$ for x, then $F_{n+k}(x) = F_n\{F_k(x)\}$.

But $F_{n+k}(x) = F_k(x) + n\frac{dF_k(x)}{dk} + \frac{n^2}{\lfloor 2}\frac{d^2F_k(x)}{dk^2} + \&c$

and $F_n\{F_k(x)\} = F_k(x) + n f_1\{F_k(x)\} + n^2 f_2\{F_k(x)\} + \&c.$

Equating the coeff^{ts} of n we have $\frac{dF_k(x)}{dk} = f_1\{F_k(x)\}$

Let $F_k(x) = y$ and $F_n(y) = z$ then we have

$\frac{dy}{dk} = f_1(y)$ $\therefore \frac{dz}{dk} = f_1(y)\frac{dz}{dy}$

$\therefore \frac{dF_n(y)}{dk} = f_1(y)\frac{dF_n(y)}{dy}$ or $\frac{dF_n(x)}{dn} = f_1(x)\frac{dF_n(x)}{dx}$

Equating the coeff^{ts} of n^{n+1} we have $n f_n(x) = f_1(x) f'_{n-1}(x)$

Cor 1. If $f_n(x) = \frac{(x)^n}{\lfloor n}\{\psi_1(n)x - \psi_2(n)x^2 + \psi_3(n)x^3 - \&c\}$, then

i. $n\psi_n(n) = n\psi_1(n-1)\psi_n(1) + (n+1)\psi_2(n-1)\psi_{n-1}(1) +$

$(n+2)\psi_3(n-1)\psi_{n-2}(1) + (n+3)\psi_4(n-1)\psi_{n-3}(1) + \&c$

ii. $\psi_n(n) = [illegible]\left\{\frac{\psi_1(n-1)}{n} - \frac{\psi_2(n-2)}{n^2} + \frac{\psi_3(n-3)}{n^3} - \&c\right\}$

[illegible] $f_1(x) f'_{n-1}(x)$ equating the coeff^{ts} of like powers

[illegible] is the coeff^{t} of x^n in $F_n(x)$ by I expanding

$$\frac{1}{4n^2} + \frac{1}{n^2+(n+1)^2} + \frac{1}{n^2+(n+2)^2} + \frac{1}{n^2+(n+3)^2} + \&c$$

$$= \frac{\pi}{4n} + \frac{1}{8\pi n^3} - \frac{\pi}{n}\cdot\frac{1}{e^{4\pi n} - 2e^{2\pi n}\cos 2\pi n + 1}$$

$$+ 4n\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^4+4n^4} + \frac{2}{e^{4\pi}-1}\cdot\frac{1}{2^4+4n^4} + \frac{3}{e^{6\pi}-1}\cdot\frac{1}{3^4+4n^4} + \&c\right\}$$

$$\frac{1}{2n^2} + \frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \&c \text{ ad inf.}$$

$$= \frac{\pi}{2n} + \frac{\pi}{n}\cdot\frac{1}{e^{2\pi n}-1}.$$

$$l\left\{\frac{1}{2(l^2+n^2)} + \frac{1}{l^2+(n+1)^2} + \frac{1}{l^2+(n+2)^2} + \&c\right\}$$

$$= \tan^{-1}\frac{l}{n} + \frac{B_2}{2}\cdot\frac{\sin\left(2\tan^{-1}\frac{l}{n}\right)}{l^2+n^2} - \frac{B_4}{4}\cdot\frac{\sin\left(4\tan^{-1}\frac{l}{n}\right)}{(l^2+n^2)^2}$$

$$+ \frac{B_6}{6}\cdot\frac{\sin\left(6\tan^{-1}\frac{l}{n}\right)}{(l^2+n^2)^3} - \&c$$

: Again find the coeff^{t}. of x^n by II expansion
& equate the two results

C. 2 (r+1) $\psi_2(1)$ = $\frac{1}{2}\psi_1(1) + \frac{B_2}{2}\psi_2\left(\frac{1}{2}\right) - \frac{B_4}{2^3}\psi_4\left(\frac{1}{4}\right)$

$+ \frac{B_6}{2^5}\psi_6\left(\frac{1}{6}\right) - \&c.$

Sol. Equate the coeff^{ts}. of x^n in V 2 Cor 3.

4. $f_1(x) = (1+x) f_1\{\log_e(1+x)\}$

Sol. In V 1 write $\log_e(1+x)$ for x; then $F_{n,1}(x) = \log_e(1+x)$

$+ n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c$

$\therefore e^{F_{n,1}(x)} = (1+x)\, e^{n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c}$

But $e^{F_{n,1}(x)} = 1 + F_n(x) = 1 + x + n f_1(x) + n^2 f_2(x) + \&c$

$\therefore (1+x) + n(1+x) f_1\{\log_e(1+x)\} + \&c = (1+x) + n f_1(x) + \&c$

Equating the coeff^{ts} of n: $f_1(x) = (1+x) f_1\{\log_e(1+x)\}$

5. i The sum of the coeff^{ts}. of the odd terms }
+ the sum of the coeff^{ts}. of the even terms } in $\phi_n(x) = \frac{1}{n}$

ii The sum of the coeff^{ts}. of the odd terms }
− the sum of the coeff^{ts}. of the even terms } in $\phi_n(x) = \frac{1}{\lfloor n}$

Sol. $f(x) = e^x - 1$ and $F_{-1}(x) = \log_e(1+x)$ Equate the coeff^{ts}.

iii. $f_1(x) = \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{48} - \frac{x^5}{180} + \frac{11x^6}{8640} - \frac{x^7}{6720} + \&c$

iv. $\phi_1(x) = x$ $\phi_2(x) = x\left(x - \frac{1}{2}\right)$

$\phi_3(x) = x\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$

$$(1+x)^n = 1 + nx(1+x)^{\frac{n-1}{2}} + \frac{n(n^2-1)}{4\lfloor 3}x^3(1+x)^{\frac{n-3}{2}} + \frac{n(n^2-1)(n^2-9)}{4^2\lfloor 5}x^5(1+x)^{\frac{n-5}{2}} + \&c$$

$$\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = 1 + nx(1+x)^{\frac{n-3}{2}} + \frac{n(n-5)(n-7)}{4\lfloor 3}x^3(1+x)^{\frac{n-9}{2}} + \frac{n(n-7)(n-9)(n-11)(n-13)}{4^2\lfloor 5}x^5(1+x)^{\frac{n-15}{2}}$$

$$\frac{1+(1+x)^n}{2} = (1+x)^{\frac{n}{2}} + \frac{n^2}{4\lfloor 2}x^2(1+x)^{\frac{n-2}{2}} + \frac{n^2(n^2-2^2)}{4^2\lfloor 4}x^4(1+x)^{\frac{n-4}{2}} + \&c.$$

$$\frac{1}{2} + \frac{1}{2}\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = (1+x)^{\frac{n}{2}} + \frac{n(n-4)}{4\lfloor 2}x^2(1+x)^{\frac{n-6}{2}} + \frac{n(n-6)(n-8)(n-10)}{4^2\lfloor 4}x^4(1+x)^{\frac{n-12}{2}} + \&c.$$

$\phi_1(2n) = 1$

$\phi_2(2n) = n$

$\phi_3(2n) = n^2 - \frac{n}{6}$

$\phi_4(2n) = n^3 - \frac{5n^2}{12} + \frac{n}{24}$

$\phi_5(2n) = n^4 - \frac{13n^3}{18} + \frac{n^2}{6} - \frac{n}{90}$

$\phi_6(2n) = n^5 - \frac{77n^4}{72} + \frac{89n^3}{216} - \frac{91n^2}{1440} + \frac{11n}{4320}$

$\phi_7(2n) = n^6 - \frac{29n^5}{20} + \frac{175n^4}{216} - \frac{149n^3}{720} + \frac{91n^2}{4320} - \frac{n}{3360}$

6. $\psi_1(n) = 1$

$\psi_2(n-1) = \frac{n}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)$

$\psi_3(n-2) = \frac{n(n-1)}{72}\left\{\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)^2 - \left(\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) - \frac{1}{n} + \frac{1}{2}\right\}$

Cor. If $\frac{x}{1-nx} = y$ and $(1-nx) = z$, then

$F_{2,4}(x) = y + \frac{y^2}{6}\log_e z + \frac{y^3}{72}\left\{(\log_e z)^2 + (1-\log_e z)^2 - z\right\} + \&c$

Sol. Apply the above results in V. 1.

Ex. Show that $f_1(x)\, f_1''(x) = f_1(x) - f_2(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$

Sol. From V. 2 Cor 3 we have $f_1'(x) = x - \frac{1}{2}f_1(x) + B_2 f_2(x)$
$- B_4 f_4(x) + B_6 f_6(x) - \&c$. Differentiating both sides and multiplying the results by $f_1(x)$ we have

$f_1(x)\, f_1''(x) = f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + B_2 f_1(x) f_2'(x) - B_4 f_1(x) f_4'(x) + \&c$

$= f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$ by V. 3.

7. $1 \cdot \Sigma 1 + \frac{1}{2}\Sigma\frac{1}{2} + \frac{1}{3}\Sigma\frac{1}{3} + \cdots + \Sigma\frac{1}{x} = (x+1)\Sigma\frac{1}{x} - x$.

[illegible] $+ \cdots + \left(\Sigma\frac{1}{x}\right)^2 = (x+1)\left(\Sigma\frac{1}{x}\right)^2 - (2x+1)\Sigma\frac{1}{x} + 2x$.

[illegible] $+ \cdots + \left(\Sigma\frac{1}{x}\right)^3 = (x+1)\left(\Sigma\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{2}\right)\left(\Sigma\frac{1}{x}\right)^2$
[illegible] $+ \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{x^2}\right)$.

$$c + \log x + \frac{x}{|\underline{1}} + \frac{x^2}{2|\underline{2}} + \frac{x^3}{3|\underline{3}} + \&c$$

$$= e^x\left(\frac{1}{x} + \frac{|\underline{1}}{x^2} + \frac{|\underline{2}}{x^3} + \cdots + \frac{|\underline{n-1}}{x^n}\theta\right)$$

$$\text{where } \theta = \frac{2}{3} + \frac{4}{135x} + \frac{8}{27.105x^2} - \&c$$

$$\text{Sf} \int_0^{x(1-h)} \frac{1-e^{-x}}{x}\,dx = c + \log_e x$$

$$\text{then } h(e^x - 1) + \frac{h^2}{2}(e^x - 1 - x) + \frac{h^3}{3}\left(e^x - 1 - x - \frac{x^2}{|\underline{2}}\right) + \&c$$

$$= \frac{1}{x} - \frac{|\underline{1}}{x^2} + \frac{|\underline{2}}{x^3} - \frac{|\underline{3}}{x^4} + \&c$$

CHAPTER VI

1. If $f(x+h) - f(x) = h\,\phi'(x)$, then

$$f(x) = \phi(x) - \frac{h}{2}\phi'(x) + \frac{B_2}{2!}h^2\phi''(x) - \frac{B_4}{4!}h^4\phi^{iv}(x) + \frac{B_6}{6!}h^6\phi^{vi}(x) - \&c.$$

2. If $f(x+h) + f(x) = h\,\phi'(x)$ then

$$f(x) = \frac{h}{2}\phi'(x) - (2^2-1)B_2\frac{h^2}{2!}\phi''(x) + (2^4-1)B_4\frac{h^4}{4!}\phi^{iv}(x) - \&c$$

Sol. If we write e^x for $\phi(x)$, we see that the coeff^{ts} in R.H.S of VI 1 are the coeff^{ts} in the expansion of $\frac{h}{e^h-1}$

Again, if we write e^x for $\phi(x)$ in VI 2 we see the coeff^{ts} in VI 2 are the coeff^{ts} in the expansion of $\frac{h}{e^h+1}$ or

$$\frac{h}{e^h-1} - \frac{2h}{e^{2h}-1}$$

3. Let $F_n(x) = \phi(x) - \frac{n-1}{n+1}\{\phi(x+h)+\phi(x-h)\} +$

$$\frac{(n-1)(n-3)}{(n+1)(n+3)}\{\phi(x+2h)+\phi(x-2h)\} - \frac{(n-1)(n-3)(n-5)}{(n+1)(n+3)(n+5)}$$

$\{\phi(x+3h)+\phi(x-3h)\} + \&c$, then

i. If $f(x+h) - f(x-h) = h\,\phi'(x)$, then

$$f(x) = \frac{F_1(x)}{1} + \frac{F_3(x)}{3} + \frac{F_5(x)}{5} + \frac{F_7(x)}{7} + \&c$$

ii. If $f(x+h) + f(x-h) = 2\phi(x)$, then

$$f(x) = F_1(x) + \frac{1}{2!}F_3(x) + \frac{1\cdot3}{2!}F_5(x) + \frac{1\cdot3\cdot5}{3!}F_7(x) + \&c$$

4. If $f(x+h) + K f(x) = \phi(x)$

Let $f(x) = \frac{\phi(x)\,\psi_0(K)}{K+1} - \frac{h}{1!}\frac{\phi'(x)\,\psi_1(K)}{(K+1)^2} + \frac{h^2}{2!}\frac{\phi''(x)\,\psi_2(K)}{(K+1)^3} - \&c$

then $\frac{1}{e^x+K} = \frac{\psi_0(K)}{K+1} - \frac{x}{1!}\frac{\psi_1(K)}{(K+1)^2} + \frac{x^2}{2!}\frac{\psi_2(K)}{(K+1)^3} - \&c$

Sol. Let $\phi(x) = e^x$ then $f(x) = \frac{e^x}{e^h+K}$

$$\int \phi(x) e^{-nx} dx = -e^{-nx}\left\{\frac{\phi(x)}{n} + \frac{\phi'(x)}{n^2} + \frac{\phi''(x)}{n^3} + \&c\right\}$$

$$\int \phi(x) \cos nx \, dx = \sin nx \left\{\frac{\phi(x)}{n} - \frac{\phi''(x)}{n^3} + \&c\right\}$$
$$+ \cos nx \left\{\frac{\phi'(x)}{n^2} - \frac{\phi'''(x)}{n^4} + \&c\right\}$$

$$\int_x^\infty e^{-x^2} \cos 2nx \, dx$$
$$= e^{-x^2}\left\{\frac{\cos(2nx+\theta)}{2r} - \frac{1.\cos(2nx+3\theta)}{2^2 r^3} + \frac{1.3\cos(2nx+5\theta)}{2^3 r^5}\right.$$
$$\left. - \frac{1.3.5\cos(2nx+7\theta)}{2^4 r^7} + \&c\right\}$$

where $\tan\theta = \frac{n}{x}$ & $r = \sqrt{n^2+x^2}$

$$\int_0^\infty e^{-x^2}\left\{e^{2nx}\phi(x) + e^{-2nx}\phi(-x)\right\} dx$$
$$= \sqrt{\pi}\, e^{n^2}\left\{\phi(n) + \frac{\phi''(n)}{4} + \frac{\phi^{iv}(n)}{4.8} + \frac{\phi^{vi}(n)}{4.8.12} + \&c\right\}$$
$$= \int_0^\infty e^{n^2-x^2}\left\{\phi(n+x) + \phi(n-x)\right\}$$

$$\int_1^\infty e^{-\frac{1}{x^2}}\left\{A_1 - \frac{x^2}{2} A_3 + \frac{x^4}{12} A_5 - \&c\right\} dx$$
$$= \sqrt{\frac{\pi}{2}}\left\{A_0 - \frac{1}{2} A_1 + \frac{2^2}{14} A_2 - \frac{2^3}{12} A_3 + \&c\right\}$$

Cor. $1^n - 2^n K + 3^n K^2 - 4^n K^3 + 5^n K^4 - \&c = \frac{\psi_n(K)}{(K+1)^{n+1}}$

Sol. $\frac{1}{e^x+K} = e^{-x} - K e^{-2x} + K^2 e^{-3x} - K^3 e^{-4x} + \&c$

$= \frac{\psi_0(K)}{K+1} - \frac{x}{\underline{1}} \frac{\psi_1(K)}{(K+1)^2} + \frac{x^2}{\underline{2}} \frac{\psi_2(K)}{(K+1)^3} - \&c$ by VI 4.

Equate the coeff^ts. of x^n.

Cor 2. $\psi_n(K) - n \frac{\psi_{n-1}(K)}{K+1} + \frac{n(n-1)}{\underline{2}} \frac{\psi_{n-2}(K)}{(K+1)^2} - \dots + (-1)^n \frac{\psi_0(K)}{(K+1)^n}$

$= (-1)^{n+1} \frac{K\, \psi_n(K)}{(K+1)^n}$

Sol. Multiply both sides in VI 4 by $e^x + K$, then we see that the coeff^t. of $x^n = 0$

Cor 3. If $\psi_n(K) = F_1(n) - F_2(n) K + F_3(n) K^2 - F_4(n) K^3 + \&c$

$\dots\dots + (-1)^{n+1} F_n(n) K^{n-1}$ then 1. $F_{n-r}(n) = F_{r+1}(n)$

ii. $F_r(n+1) + n F_{r-1}(n-1) + \frac{n(n+1)}{\underline{2}} F_{r-2}(n-1) + \dots + \frac{\underline{n+r-1}}{\underline{r}\,\underline{n-1}} F_1(n)$

$= r^n$ Sol. Equate the coeff^ts. of K^{r-1} in VI 4. cor 1

Cor 4. $F_r(n+1) = r^{n+1} - n(r-1)^{n+1} + \frac{n(n-1)}{\underline{2}} (r-2)^{n+1} - \&c$ to r terms

Sol. Multiply both sides in VI 4 Cor. by $(1+K)^{n+1}$ and then equate the coeff^ts. of K^{r-1}

Cor 5. $\psi_n(-1) = \underline{n}$, $\psi_n(1) = 2^{n+1}(2^{n+1}-1) \frac{B_{n+1}}{n+1} \sin\frac{\pi n}{2}$, $\psi_0(K) = 1$

$\psi_1(K) = 1$

$\psi_2(K) = 1 - K$

$\psi_3(K) = 1 - 4K + K^2$

$\psi_4(K) = 1 - 11K + 11K^2 - K^3$

$\psi_5(K) = 1 - 26K + 66K^2 - 26K^3 + K^4$

$\psi_6(K) = 1 - 57K + 302K^2 - 302K^3 + 57K^4 - K^5$

$\psi_7(K) = 1 - 120K + 1191K^2 - 2416K^3 + 1191K^4 - 120K^5 + K^6$

Write under each term the sum of the product of its coeff^t. and the no. of terms from the left and the product of the coeff^t. of the preceding term and its no. of terms from [illegible]

i. $B_{2n} = I_{2n} + (-1)^n (F_{2n} - 1)$

where I_{2n} is ~~the nearest~~ integer to B_{2n} and F_{2n} is the sum of the reciprocals of prime no.s next to the factors of $2n$ including unity and the number itself.

ii. The numerator of B_n is divisible by the greatest odd multiple of n prime to $(2^n - 1)$.

$I_0 = I_2 = I_4 = I_6 = I_8 = I_{10} = I_{12} = 0$.

$I_{14} = 1$, $I_{16} = 7$, $I_{18} = 55$, $I_{20} = 529$.

$I_{22} = 6192$, $I_{24} = 86580$, $I_{26} = 1425517$

Cor. $\psi_n(x-1)$ is the integral part of

$$\frac{x^{n+1}}{1-x}\left\{\frac{\lfloor n}{\left(\log_e\frac{1}{1-x}\right)^{n+1}} - \frac{B_{n+1}}{n+1}\sin\frac{n\pi}{2}\right\}$$

Sol. $e^{-x} + e^{-2x} + e^{-3x} + \&c = \frac{1}{x} -$ [illegible]

Differentiating n times we have

$1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c = \frac{\lfloor n}{x^{n+1}} \pm \&c$

writing $\log_e\frac{1}{1-x}$ for x we have

$$\frac{1^n}{1-x} + \frac{2^n}{(1-x)^2} + \frac{3^n}{(1-x)^3} + \&c = \frac{\lfloor n}{\left(\log_e\frac{1}{1-x}\right)^{n+1}} \pm \&c$$

Applying VI & Cor 1. we at once get the result.

Ex. 1. Shew that $f(x)$ is the term independent of n in

$$\frac{\phi(x) + \frac{1}{n}\phi'(x) + \frac{1}{n^2}\phi''(x) + \frac{1}{n^3}\phi'''(x) + \&c}{e^{n-k} + k}$$

2. If $n > 1$, $F_n(n-1)$ is the coeff. of $\frac{x^{n-1}}{\lfloor n-1}$ in $e^{x(n-1)}$ [illegible]

3. $\frac{\cos x + k}{1 + 2k\cos x + k^2} = \frac{\psi_0(k)}{k+1} - \frac{x^2}{\lfloor 2}\frac{\psi_2(k)}{(k+1)^3} + \frac{x^4}{\lfloor 4}\frac{\psi_4(k)}{(k+1)^5} - \&c$

4. $\frac{\sin x}{1 + 2k\cos x + k^2} = \frac{x}{\lfloor 1}\frac{\psi_1(k)}{(k+1)^2} - \frac{x^3}{\lfloor 3}\frac{\psi_3(k)}{(k+1)^4} + \frac{x^5}{\lfloor 5}\frac{\psi_5(k)}{(k+1)^6}$

5. If n is even shew that $\psi_n(k)$ is divisible by $(1-k)$

6. [illegible] $(S_2-1) - 2^n(S_3-1) + 3^n(S_4-1) - \&c = \cos n\pi -$ [illegible]

[illegible] $\frac{n\pi}{2}\frac{B_{n+1}}{n+1}(2^{n+1}-1)$ [illegible] S_2 [illegible] &c where

$S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ then

[illegible] $\frac{n(n-1)}{\lfloor 2}A_{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3}A_{n-3}$ [illegible] &c + [illegible]

$+ \frac{n(n-1)}{\lfloor 2}(n-2)^n - \&c$

[illegible] x^n in $(e^x$ [illegible]$)^n$

$$\frac{3392780147 + 6960\left\{\frac{1^{27}x}{1-x} + \frac{2^{27}x^2}{1-x^2} + \frac{3^{27}x^3}{1-x^3} + \&c\right\}}{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \&c\right)}$$

$$= 489693897\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \&c\right)\right\}^6$$

$$+ 2507636250\left\{1 + 240\left(\frac{1^3 x}{1-x} + \&c\right)\right\}^3\left\{1 - 504\left(\frac{1^5 x}{1-x} + \&c\right)\right\}^2$$

$$+ 395450000\left\{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \&c\right)\right\}^4.$$

$$B_{38} = \frac{2929,993913,841559}{6}.$$

~~$$\frac{17231682552\overline{01} - 171864\left\{\frac{1^{29}x}{1-x} + \frac{2^{29}x^2}{1-x^2} + \frac{3^{29}x^3}{1-x^3} + \&c\right\}}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \&c\right)}.$$~~

$\ldots(K\ldots) = A_n - A_{n-1}K + A_{n-2}K^2 - \&c$ to n terms.

7. Show that

$\frac{1^5}{1} + \frac{2^5}{2^2} + \ldots + \frac{\ldots}{2^5} + \&c = 1082.$

$\frac{1^5}{3} + \frac{2^5}{3^2} + \frac{3^5}{3^3} + \&c \ldots = \frac{\ldots}{4}.$

5. $\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_2\frac{x^2}{\lfloor 2} - B_4\frac{x^4}{\lfloor 4} + B_6\frac{x^6}{\lfloor 6} - \&c$

Sol. write e^x for $\phi(x)$ in VI.

Cor 1. $\frac{x}{e^x+1} = \frac{x}{2} - B_2\frac{x^2}{\lfloor 2}(2^2-1) + B_4\frac{x^4}{\lfloor 4}(2^4-1) - \frac{B_6}{\lfloor 6}x^6(2^6-1) + \&c$

Sol. $\frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1}$

Cor 2. $\log_e \frac{x}{e^x-1} = -\frac{x}{2} - B_2\frac{x^2}{2\lfloor 2} + B_4\frac{x^4}{4\lfloor 4} - B_6\frac{x^6}{6\lfloor 6} + \&c$

Sol. $\log_e(e^x - 1) = \int \frac{e^x}{e^x-1}\,dx$

Cor 3. $\log_e \frac{2}{e^x+1} = -\frac{x}{2} - B_2\frac{x^2}{2\lfloor 2}(2^2-1) + B_4\frac{x^4}{4\lfloor 4}(2^4-1)$ &c

Sol. $\log_e(e^x+1) = \log_e(e^{2x}-1) - \log_e\ldots$

Ex. If P, Q, R, S &c are so small that $\frac{1}{720}$ of the sum of their cubes may be neglected, show that

1. If $e^P + e^Q + e^R = 2 + e^{P+Q+R}$ then

$\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{2} + \frac{P+Q+R}{12} = 0$

2. If $e^{P+Q+R+S} = \frac{e^P + e^Q + e^R + e^S - 2}{e^{-P} + e^{-Q} + e^{-R} + e^{-S} - 2}$ then

$\ldots + \frac{1}{2} \ldots \frac{P+Q+R+S}{12}$

$$B_{22} = \frac{11(57183 + 20500)}{138}$$

$$B_{24} = \frac{236364091}{2730} = \frac{19.1617^2 + 10.4200^2 + 34.550^2}{2730}$$

$$B_{26} = \frac{8553103}{6} = \frac{13(392931 + 265000)}{6}$$

$$236364091 + 131040\left(\frac{1^{23}x}{1-x} + \frac{2^{23}x^2}{1-x^2} + \&c\right)$$
$$6$$

$$= 49679091\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^6$$

$$+ 176400000\left\{1 + 240\left(\frac{1^3x}{1-x} + \&c\right)\right\}^3\left\{1 - 504\left(\frac{1^5x}{1-x} + \&c\right)\right\}^2$$

$$+ 10285000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^4.$$

$$B_{28} = \frac{23749461029}{870} = \frac{7}{870}\left(19.23.11^2.21^3 + 2.525^2.4549 + 55.10^4.719\right)$$

$$B_{30} = \frac{8615841276005}{14322}.$$

$$B_{32} = \frac{7709321041217}{510}$$

$$B_{34} = \frac{2577687858367}{6}.$$

$$B_{36} = \frac{26,315271,553053,477373}{1919190}$$

5. $2^{P+Q+R+S+T}$ ...

$(e^P+e^R+e^R+e^S+e^T-2)^2 - (e^{2P}+e^{2Q}+e^{2R}+e^{2S}+e^{2T}-2)$, then

$e^P + e^Q + e^R + e^S + e^T = 2$

$\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+\frac{1}{T}+\frac{P+Q+R+S+T}{2}=\frac{1}{2}$

6. $x\cot x = 1 - B_2\frac{(2x)^2}{\lfloor 2} - B_4\frac{(2x)^4}{\lfloor 4} - B_6\frac{(2x)^6}{\lfloor 6} - B_8\frac{(2x)^8}{\lfloor 8}$ &c.

Sol. ... $x i$ for x in VI 5

N.B. From the nature of the coeff^ts we see that $B_0 = 1$

Cor. $(2n+1)B_{2n} = 2B_2B_{2n-2}\frac{\lfloor 2n}{\lfloor 2 \lfloor 2n-2}$ + $2B_4B_{2n-4}\frac{\lfloor 2n}{\lfloor 4 \lfloor 2n-4}$ + &c the last term being $2B_{n-1}B_{n+1}$ ... or $(B_n)^2\frac{\lfloor 2n}{(\lfloor n)^2}$ according as n is odd or even

Sol. $\cot^2 x = -(1+\frac{d\cot x}{dx})$ & equate the coeff^ts of x^{2n}

$B_0 = 1$, $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = \frac{1}{30}$, $B_{10} = \frac{5}{66}$,

$B_{12} = \frac{691}{2730}$, $B_{14} = \frac{7}{6}$, $B_{16} = \frac{3617}{510}$, $B_{18} = \frac{43867}{798}$, $B_{20} = \frac{174611}{330}$, B_{22} ...

7. 1. $x\,\mathrm{cosec}\,x = 1 + B_2\frac{x^2(2^2-2)}{\lfloor 2} + B_4\frac{x^4(2^4-2)}{\lfloor 4}$ + &c.

Sol. $\mathrm{cosec}\,x = \cot\frac{x}{2} - \cot x$

2. $x\tan x = B_2\frac{2^2-1}{\lfloor 2}(2x)^2 + B_4\frac{2^4-1}{\lfloor 4}(2x)^4$ + &c.

Sol. $\tan x = \cot x - 2\cot 2x$

3. $\log(x\,\mathrm{cosec}\,x) = B_2\frac{(2x)^2}{2\lfloor 2} + B_4\frac{(2x)^4}{4\lfloor 4} + B_6\frac{(2x)^6}{6\lfloor 6}$ + &c.

... $\cot x\,dx$

...

...

... $\cos x$ ...

If $\alpha, \beta, \gamma, \delta$ &c are the roots of the eqn $f(x)=0$
then $f(x) = f(0)\left(1-\frac{x}{\alpha}\right)\left(1-\frac{x}{\beta}\right)\left(1-\frac{x}{\gamma}\right)$ &c

$$\pi x \left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\} \&c$$

$$= \left(\frac{\lfloor n}{x^n}\right)^2 \sqrt{\frac{\cosh 2\pi x - \cos 2\pi x}{2}}\,^{*}\, e^{-\frac{S_2}{x^2}+\frac{S_4}{2x^4}-\frac{S_6}{3x^6}+\&c}$$

where $S_p = 1^p + 2^p + 3^p + \cdots n^p$.

$$\pi (x^2+n^2)^{n+\frac{1}{2}} \left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\} \&c$$

$$= \left(\lfloor n\right)^2 \sqrt{\frac{\cosh 2\pi x - \cos 2\pi x}{2}}\,^{*}\, e^{2n - 2x\tan^{-1}\frac{x}{n} - \frac{B_2 S_2}{x} - \frac{B_4 S_4}{2x^3} - \frac{B_6 S_6}{3x^5} - \&c}$$

where $S_p = \frac{n}{x} - \frac{p(p+1)}{\lfloor 3}\cdot\left(\frac{n}{x}\right)^3 + \frac{p(p+1)(p+2)(p+3)}{\lfloor 5}\times\left(\frac{n}{x}\right)^5 - \&c$

$$2\pi (n^2+x^2)^{n-\frac{1}{2}} \left\{1+\left(\frac{x}{n}\right)^2\right\}\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}$$

$$= \left(\lfloor n-1\right)^2 e^{2n + 2x\theta - \frac{2 B_2 \cos\theta}{1.2 r} + \frac{2 B_4 \cos 3\theta}{3.4 r^3} - \&c} \times \left(1 - e^{-2\pi x} E.\right)$$

where $r^2 = n^2 + x^2$, $\tan\theta = \frac{x}{n}$ & E the error is less than 1 & = $\cos 2\pi n$ when x is [illegible]

7. $\sec x = E_1 + \frac{x^2}{\underline{|2}} E_3 + \frac{x^4}{\underline{|4}} E_5 + \frac{x^6}{\underline{|6}} E_7 + \&c$

Cor. $\frac{B_{2n}}{2n} 2^{2n}(2^{2n}-1) = 2E_1 E_{2n-1} + 2E_3 E_{2n-3} \frac{(2n-2)(2n-3)}{\underline{|2}} + \&c$ the last term being $2E_{n-1}E_{n+1}\frac{\underline{|2n-2}}{\underline{|n-2}\,\underline{|n}}$ or $(E_n)^2 \frac{\underline{|2n-2}}{(\underline{|n-1})^2}$ according as n is even or odd

Sol. $\frac{d\tan x}{dx} = \sec^2 x$. Equate the coeff^ts of x^{2n-2}

$E_1 = 1$, $E_3 = 1$, $E_5 = 5$, $E_7 = 61$, $E_9 = 1385$, $E_{11} = 50521$,

$E_{13} = 2702765$, $E_{15} = 199360981$, &c

8\. 1. $\frac{\sin x}{x} = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2^2\pi^2})(1 - \frac{x^2}{3^2\pi^2})$ &c

Sol. The roots of the equation $\frac{\sin x}{x} = 0$ are $\pm\pi, \pm 2\pi$, &c and $\frac{\sin x}{x}$ [illegible]

2. In a similar manner

$\cos x = (1 - \frac{4x^2}{\pi^2})(1 - \frac{4x^2}{3^2\pi^2})(1 - \frac{4x^2}{5^2\pi^2})(1 - \frac{4x^2}{7^2\pi^2})$ &c

Cor 1. $\frac{e^x - e^{-x}}{2x} = (1 + \frac{x^2}{\pi^2})(1 + \frac{x^2}{2^2\pi^2})(1 + \frac{x^2}{3^2\pi^2})$ &c

2. $\frac{e^x + e^{-x}}{2} = (1 + \frac{4x^2}{\pi^2})(1 + \frac{4x^2}{3^2\pi^2})(1 + \frac{4x^2}{5^2\pi^2})$ &c

3. $\cos x + \sin x = (1 + \frac{4x}{\pi})(1 - \frac{4x}{3\pi})(1 + \frac{4x}{5\pi})$ &c

Ex 1. $\frac{\sin(x+a)}{\sin a} = (1 + \frac{x}{a})(1 - \frac{x}{\pi - a})(1 + \frac{x}{\pi + a})(1 - \frac{x}{2\pi - a}) \times$

$(1 + \frac{x}{2\pi + a})(1 - \frac{x}{3\pi - a})(1 + \frac{x}{3\pi + a})$ &c

2. $\frac{\cos(x+a)}{\cos a} = (1 + \frac{x}{\frac{\pi}{2} + a})(1 - \frac{x}{\frac{\pi}{2} - a})(1 + \frac{x}{\frac{3\pi}{2} + a}) \times$

$(1 - \frac{x}{\frac{3\pi}{2} - a})$ &c

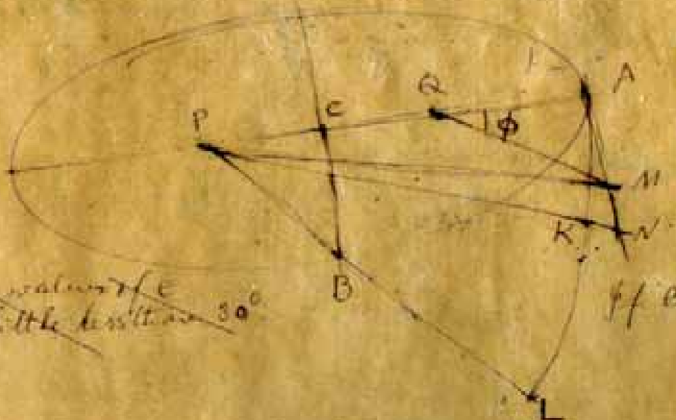
[illegible] $= (1 - \frac{x}{a})(1 + \frac{x}{\pi - a})(1 - \frac{x}{\pi + a})(1 + \frac{x}{2\pi - a})$ &c

$$\int_0 \frac{\sin x}{x}\, dx = \frac{\pi}{2} - r\cos(x-\theta).$$

$$\int_0 \frac{1-\cos x}{x}\, dx = C + \log x - r\sin(x-\theta).$$

where $r^2 = \frac{\lfloor 1}{x^2} - \frac{\lfloor 3}{2x^4} + \frac{\lfloor 5}{3x^6} - \frac{\lfloor 7}{4x^8} + \&c$

$$\left.\begin{array}{l} r\cos\theta = \frac{1}{x} - \frac{\lfloor 2}{x^3} + \frac{\lfloor 4}{x^5} - \&c \\ r\sin\theta = \frac{\lfloor 1}{x^2} - \frac{\lfloor 3}{x^4} + \frac{\lfloor 5}{x^6} - \&c \end{array}\right\}$$



$\left\{\begin{array}{l} \angle MPN = \frac{1}{2}\angle APN \\ CP = CB = CQ \end{array}\right.$

$$\frac{AL}{AK} = \frac{AB}{AN}$$

~~For other values of e φ is a little less than 30°~~

If $e = 0{\cdot}01$ $\phi = 30^\circ$

$$\pi(a+b)\left\{1 + \frac{\sin^2\theta}{2+\cos^2\frac{\theta}{2}}\right\} \text{ where } \sin\theta = \frac{a-b}{a+b}\sin\phi.$$

when $e = 1$ $3\phi - \pi = 0^\circ - 14' - 46''$

ϕ rapidly diminishes to 60°.

$$\pi(a+b)\left\{1 + 4\sin^2\frac{\theta}{2}\right\} \text{ where } \sin\theta = \frac{a-b}{a+b}\sin\phi$$

when $e = 1$ $3\phi - \frac{\pi}{2} = 0^\circ - 52' - 19''$

4. $\frac{\sin x}{\cos a} = \left(1+\frac{x}{\frac{\pi}{2}-a}\right)\left(1+\frac{x}{\frac{\pi}{2}+a}\right)\left(1-\frac{x}{\frac{3\pi}{2}-a}\right)\left(1+\frac{x}{\frac{3\pi}{2}+a}\right)$ &c

N.B. If we write the above results as

Ex. $\phi(x) = (1+a_1 x)(1+a_2 x)(1+a_3 x)$ &c, then it is possible to find $(1+a_1^n x^n)(1+a_2^n x^n)(1+a_3^n x^n)$ &c

9. $\cot x = \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x}$ &c

Sol. Equate the coeff^{ts} of x in VI 8 Ex 1.

Cor 1. $\tan x = \frac{1}{\frac{\pi}{2}-x} - \frac{1}{\frac{\pi}{2}+x} + \frac{1}{\frac{3\pi}{2}-x} - \frac{1}{\frac{3\pi}{2}+x} +$ &c

2. $\operatorname{cosec} x = \frac{1}{x} + \frac{1}{\pi - x} - \frac{1}{\pi + x} - \frac{1}{2\pi - x}$ &c

3. $\sec x = \frac{1}{\frac{\pi}{2}-x} + \frac{1}{\frac{\pi}{2}+x} - \frac{1}{\frac{3\pi}{2}-x} - \frac{1}{\frac{3\pi}{2}+x} +$ &c

10. $\tan^{-1}\frac{x}{a} - \tan^{-1}\frac{x}{\pi-a} + \tan^{-1}\frac{x}{\pi+a} -$ &c $= \tan^{-1}\left\{\frac{e^x - e^{-x}}{e^x + e^{-x}} \cot a\right\}$

Sol. $L.H.S = \frac{1}{2i}\log_e\left\{\frac{1+\frac{xi}{a}}{1-\frac{xi}{a}} \cdot \frac{1-\frac{xi}{\pi-a}}{1+\frac{xi}{\pi-a}}\right.$ &c$\left.\right\}$ Apply VI 8 Ex 1

Cor 1. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a} - \tan^{-1}\frac{x}{\frac{\pi}{2}+a} + \tan^{-1}\frac{x}{\frac{3\pi}{2}-a} -$ &c $=$

$\tan^{-1}\left\{\frac{e^x - e^{-x}}{e^x + e^{-x}} \tan k\right\}$

Cor 2. $\tan^{-1}\frac{x}{1} - \tan^{-1}\frac{x}{3} + \tan^{-1}\frac{x}{5} -$ &c $= \tan^{-1}\left(\frac{e^{\frac{\pi x}{2}} - 1}{[illegible]}\right)$

Cor 3. $\tan^{-1}\frac{x}{a} + \tan^{-1}\frac{x}{\pi-a} - \tan^{-1}\frac{x}{\pi+a} -$ &c $= \tan^{-1}\left\{\frac{e^x - e^{-x}}{2} \operatorname{cosec} a\right\}$

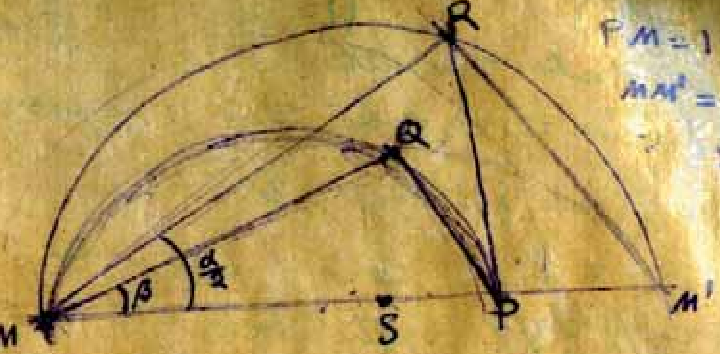
Cor 4. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a} + \tan^{-1}\frac{x}{\frac{\pi}{2}+a} - \tan^{-1}\frac{x}{\frac{3\pi}{2}-a} -$ &c $= \tan^{-1}\left\{\frac{e^x - e^{-x}}{2} \sec a\right\}$

Cor 5. $\tan^{-1}\frac{x}{1} + \tan^{-1}\frac{x}{3} - \tan^{-1}\frac{x}{5} - \tan^{-1}\frac{x}{7}$ &c $=$

$\tan^{-1}\left(\frac{e^{\frac{\pi x}{2}} - 1}{[illegible]}\right)$

Ex. [illegible] &c

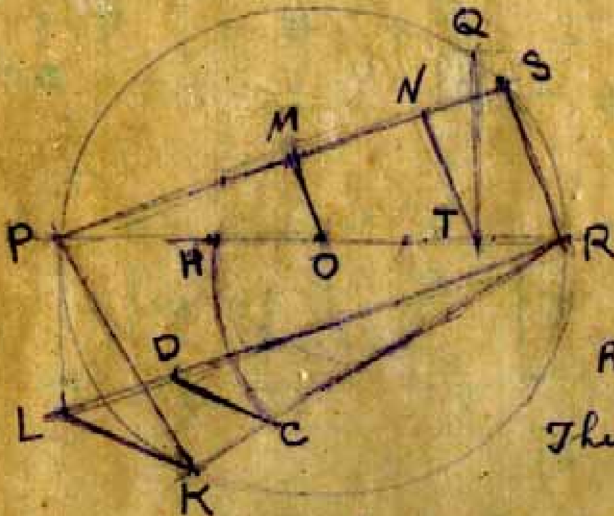
[illegible] &c



PM = 1; PQ = PM'.

MM' = m; MS = M'S.

2PS = m Cos α.



OH = ½ OP }
RT = ⅓ OR }
RS = TQ
OM, TN, RS are ∥.

PK = PM & PL = MN
RC = RH & CD is ∥[l] to RK

Then $RD^2 = \odot$

~~From RD cut off $\frac{1}{23275366}$ th of it.~~

$$\pi = \frac{355}{113}\left(1 - \frac{\cdot 0003}{3533}\right).$$

Perimeter of an ellipse $= \pi(a+b)\left\{1 + \frac{3t}{10 + \sqrt{4-3t}}\right\}$ very nearly

where $t = \left(\frac{a-b}{a+b}\right)^2$

CHAPTER VII

1. i. $\frac{1}{1^2-x^2}+\frac{1}{2^2-x^2}+\frac{1}{3^2-x^2}+\frac{1}{4^2-x^2}+\&c=\frac{1}{2x^2}-\frac{\pi}{2x}\cot\pi x.$

ii. $\frac{1}{1^2-x^2}+\frac{1}{3^2-x^2}+\frac{1}{5^2-x^2}+\frac{1}{7^2-x^2}+\&c=\frac{\pi}{4x}\tan\frac{\pi x}{2}.$

iii. $\frac{1}{1^2-x^2}-\frac{1}{2^2-x^2}+\frac{1}{3^2-x^2}-\frac{1}{4^2-x^2}+\&c=\frac{\pi}{2x}\operatorname{cosec}\pi x-\frac{1}{2x^2}$

iv. $\frac{1}{1^2-x^2}-\frac{3}{3^2-x^2}+\frac{5}{5^2-x^2}-\&c=\frac{\pi}{4}\sec\frac{\pi x}{2}.$

Ex. 1. $\frac{1}{1^2+x^2}+\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}+\&c=\frac{\pi}{2x}\,\frac{e^{\pi x}+e^{-\pi x}}{e^{\pi x}-e^{-\pi x}}-\frac{1}{2x^2}$

2. $\frac{1}{1^2+x^2}+\frac{1}{3^2+x^2}+\frac{1}{5^2+x^2}+\&c=\frac{\pi}{4x}\,\frac{e^{\pi x}-1}{e^{\pi x}+1}$

3. $\frac{1}{1^2+x^2}-\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}-\&c=\frac{1}{2x^2}-\frac{\pi}{x(e^{\pi x}-e^{-\pi x})}$

4. $\frac{1}{1^2+x^2}-\frac{3}{3^2+x^2}+\frac{5}{5^2+x^2}-\frac{7}{7^2+x^2}+\&c=\frac{\frac{\pi}{2}}{e^{\frac{\pi x}{2}}+e^{-\frac{\pi x}{2}}}$

2. i. $\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\&c=\frac{(2\pi)^{2n}}{2\,|\underline{2n}}B_{2n}$

ii. $\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\&c=\frac{(2^{2n}-1)\pi^{2n}}{2\,|\underline{2n}}B_{2n}$

iii. $\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\&c=\frac{(2^{2n}-2)\pi^{2n}}{2\,|\underline{2n}}B_{2n}$

iv. $\frac{1}{1^{2n-1}}-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\&c=\frac{\pi^{2n-1}}{2^{2n}\,|\underline{2n-1}}E_{2n-1}$

N.B. From chap. VI we know the values of B_n for even integers and E_n for odd integers, we know that B_n and E_n are infinite when $n=\infty$; that the values of B_n are all fractions though $(2^n-1)B_n$ is an integer n being even and those of E_n are all integers n being odd, in both cases n being positive. Now let us try to interpret some meaning for B_n and E_n when n is positive integer [illegible]

If $l = (a-b)\cos\phi = (a+b)\tan\theta$

and $\frac{\pi l}{\theta}$ be the perimeter of the ellipse

then $\phi = \frac{2\sqrt{ab}}{a+b}\left\{30^\circ + 6^\circ 18' 49'' \frac{(\sqrt{a}-\sqrt{b})^4}{a+b} - 1^\circ 10' 55'' \left(\frac{a-b}{a+b}\right)^2\right\}$

(ϕ in the figure).

~~$3\phi - 90^\circ = \frac{4ab}{(a+b)^2}\left(\frac{a-b}{a+b}\right)^2\left\{72^\circ 42' 3'' \cdot \frac{ab}{(a+b)^2} - 2^\circ 12' 14''\right\}$~~

$\phi = 30^\circ + h(1-h)\left\{5^\circ 19\tfrac{1}{2}' - 6^\circ 3\tfrac{1}{2}' h\right\}$

where $h = \left(\frac{a-b}{a+b}\right)^2$

N.B. Art VII 2 is true for all positive values of n. For B_n and E_n for negative values of n see Chapter IX.

3. $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+ \&c = S_n = \frac{(2\pi)^n}{2\underline{|n}} B_n$

From this we can find B_n when $n > 1$.

$B_1 = \infty$, $B_3 = \frac{3}{2\pi^3} S_3$ &c

$B_{\frac{1}{2}} = \frac{3}{4\pi\sqrt{2}} S_{\frac{1}{2}}$ &c.

N.B. To find $\underline{|n}$ for fractional values of n see chap.

4. $\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+ \&c = \frac{(2^n-1)\pi^n}{\underline{|n}} B_n$

From this we can find B_n if n is not negative.

$B_0 = -1$, $B_{\frac{1}{2}} = -\left(1+\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\&c\right)$

5. $\frac{1}{1^n}-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\&c = \frac{\pi^n}{2^{n+1}\underline{|n-1}} E_n$

From this we can find E_n if n is not negative.

$E_0 = \infty$, $E_{\frac{1}{2}} = 2\sqrt{2}\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{7}}+\&c\right)$

$E_2 = \frac{8}{\pi^2}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\&c\right)$

6. $\frac{1}{(a+b)^n}+\frac{1}{(a+2b)^n}+\frac{1}{(a+3b)^n}+\&c = \frac{1}{b(n-1)a^{n-1}}-\frac{1}{2a^n}$

$+ B_2\frac{n}{\underline{|2}}\frac{b}{a^{n+1}} - B_4\frac{n(n+1)(n+2)}{\underline{|4}}\frac{b^3}{a^{n+3}}+\&c$

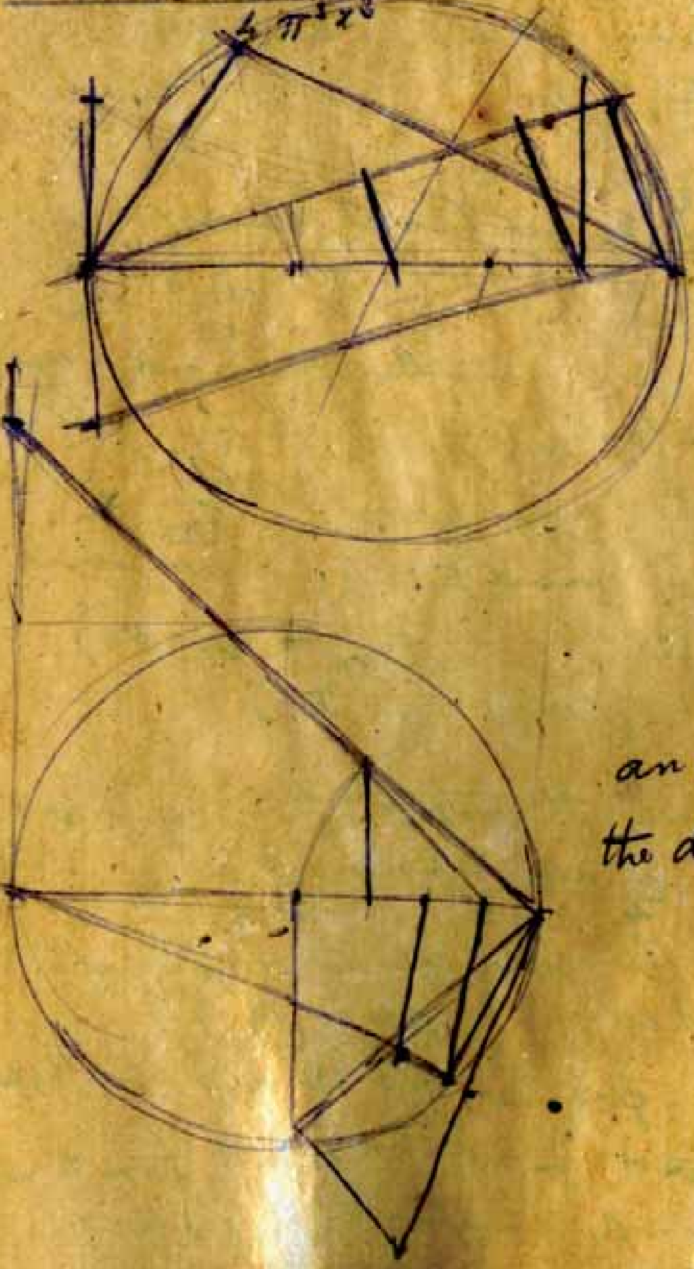
From this we can sum up the reciprocals of powers of all numbers in A.P. approximately.

Sol. Let the L.H.S. $= \phi(a)$, then $\phi(a-b)-\phi(a) = \frac{1}{a^n}$. Apply VI

Ex. $S_n = \frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots+\frac{1}{(x-1)^n}+\frac{1}{2x^n}+\frac{1}{(n-1)x^{n-1}}$

$+ \frac{B_2}{2}\frac{n}{x^{n+1}} - B_4\frac{n(n+1)(n+2)}{\underline{|4}}\frac{1}{x^{n+3}}+\&c$

$$\left(1+\frac{x^6}{1^6}\right)\left(1+\frac{x^6}{2^6}\right)\left(1+\frac{x^6}{3^6}\right)\&c = \frac{\sinh 2\pi x - 2\sinh \pi x \cos \pi x\sqrt{3}}{4\pi^3 x^3}.$$



an inch greater if the diameter be 5000 miles long

$S_2 = 1.644934066848$ $\qquad \frac{1}{B_1} = 0, \quad \frac{1}{B_2} = 6$

$S_3 = 1.2020569031$ $\qquad \frac{1}{B_3} = 17.19624$

$S_4 = 1.0823232337$

$S_5 = 1.0$[illegible] $\qquad \frac{1}{B_4} = 30, \quad \frac{1}{B_5} = 37.34953$

$S_6 = 1.0173430620$ $\qquad \frac{1}{B_6} = 42 \quad \frac{1}{B_7} = 38.03528$

$S_7 = 1.00$[illegible]774

$S_8 = 1.0040773562$ $\qquad \frac{1}{B_8} = 30 \quad \frac{1}{B_9} = 20.98719$

$S_9 = 1.0020083928$ $\qquad \frac{1}{B_{10}} = 13.2.$

$S_{10} = 1.0009945781$

Ex. 1. Show that when $n = 0$ $\left(\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \&c\right) - \frac{1}{n}$ is a finite quantity the value of which is $.5$[illegible] nearly and $n S_{n+1} = 1$

Sol. In VII 6 Cor write $n+1$ for n and 1 for x, then we have $S_{n+1} - \frac{1}{n} = \frac{1}{2} + B_2 \frac{n+1}{\lfloor 2} - \&c.$

When $n = 0$ $S_{n+1} - \frac{1}{n} = \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \&c$

$= .577$ nearly.

Show that $\pi x B_{n+1} = 1$ when $n = 0$.

Sol. We have $S_{n+1} = \frac{(2\pi)^{n+1}}{2 \lfloor n+1} B_{n+1}$. Multiply both sides by n and then writing 0 for n, we have $1 = \frac{(2\pi)^{n+1}}{2 \lfloor n+1} n B_{n+1}$ when $n = 0$

$\therefore$ when $n = 0$ $\pi x B_{n+1} = 1$

3 Show that $\frac{1 + E_2}{1 - E_2} = \frac{\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \&c}{\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{15^2} + \&c}$

Sol. [illegible] $\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right)$

$= \frac{\frac{1}{1^2} + \frac{1}{5^2} - \&c}{\frac{1}{3^2} + \frac{1}{7^2} + \&c}$ Apply Componendo & dividendo

$$\frac{1}{(1-x^2)(1-x^3)(1-x^5)(1-x^7)(1-x^{11})(1-x^{13})(1-x^{17})\ \&c}$$

$$= 1 + \frac{x^2}{1-x} + \frac{x^5}{(1-x)(1-x^2)} + \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \frac{x^{17}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \&c$$

If $f(x) = \phi(x) + \phi(2x) + \phi(3x) + \phi(4x) + \&c$

then $\phi(x) + \phi(4x) + \phi(9x) + \phi(16x) + \&c$

$$= f(x) - f(2x) - f(3x) + f(4x) - f(5x) + f(6x) - f(7x)$$

$$- f(8x) + f(9x) + f(10x) - f(11x) - f(12x) - \&c$$

If 2, 3, 5, 7, 11 &c are prime numbers, then

$\sqrt{\frac{a(a-1)}{4} + \sqrt{\frac{a(a-1)}{4} + \sqrt{\frac{a(a-1)}{4} + \&c}}}$ to n terms + h.

$$= \frac{a}{2}\left\{1 - \mu/a^n + \frac{(\mu/a^n)^2}{2(a-1)} - \frac{(\mu/a^n)^3}{2(a-1)(a^2-1)} + \frac{(\mu/a^n)^4(a+5)}{8(a-1)(a^2-1)(a^3-1)}\right.$$

$$\left. - \frac{(\mu/a^n)^5(7a^2+3a+7)}{8(a-1)(a^2-1)(a^3-1)(a^4-1)} + \&c\right\}$$

where μ is a function of a & h independent of n

$$\frac{2h}{a} = 1 - \mu + \frac{\mu^2}{2(a-1)} - \frac{\mu^3}{2(a-1)(a^2-1)} + \&c$$

6. Shew that

i. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\&c = \frac{\pi^2}{6}$

ii. $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\&c = \frac{\pi^3}{32}$

iii. $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\&c = \frac{\pi^4}{90}$

7. $\frac{1}{1-a_2}\cdot\frac{1}{1-a_3}\cdot\frac{1}{1-a_5}\cdot\frac{1}{1-a_7}\cdot\frac{1}{1-a_{11}}$ &c where 2, 3, 5, 7, 11 &c are all prime numbers

$=1+a_2+a_3+a_2a_2+a_5+a_2a_3+a_7+a_2a_2a_2+\&c$

where the suffixes are natural numbers written into prime factors.

Cor 1. $\frac{2^n}{2^n-1}\cdot\frac{3^n}{3^n-1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n-1}\cdot\frac{11^n}{11^n-1}$ &c where 2, 3, 5, 7 &c are prime numbers $=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\&c$ or S_n

Cor 2. $\frac{2^n}{2^n+1}\cdot\frac{3^n}{3^n+1}\cdot\frac{5^n}{5^n+1}$ &c $=\frac{S_{2n}}{S_n}$

Sol. $\frac{2^n}{2^n-1}\cdot\frac{3^n}{3^n-1}\cdot\frac{5^n}{5^n-1}$ &c $=S_n$

$\therefore \frac{2^{2n}}{2^{2n}-1}\cdot\frac{3^{2n}}{3^{2n}-1}\cdot\frac{5^{2n}}{5^{2n}-1}$ &c $=S_{2n}$

$\therefore \frac{2^n}{2^n+1}\cdot\frac{3^n}{3^n+1}\cdot\frac{5^n}{5^n+1}$ &c $=\frac{S_{2n}}{S_n}$

Cor 3. $\frac{2^n+1}{2^n-1}\cdot\frac{3^n+1}{3^n-1}\cdot\frac{5^n+1}{5^n-1}$ &c $=\frac{(S_n)^2}{S_{2n}}$

Cor 4. $\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{8^n}+\frac{1}{11^n}+\frac{1}{12^n}+\&c$ where 2, 3, 5, 7, 8 are natural numbers containing an odd number of prime factors

$=\frac{(S_n)^2-S_{2n}}{2S_n}$

Sol. Subtract Cor 2 from Cor 1 after applying VIII 7.

$$\frac{1}{2\pi\sqrt{3}\,x^4} + \frac{1}{1^4+1^2x^2+x^4} + \frac{2}{2^4+2^2x^2+x^4} + \frac{3}{3^4+3^2x^2+x^4} + \&c$$

$$= \frac{\pi}{3x^2\sqrt{3}} \cdot \frac{\cosh \pi x\sqrt{3} + 2\cos \pi x}{\cosh \pi x\sqrt{3} - \cos \pi x}$$

$$+ 2\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+1^2x^2+x^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4+2^2x^2+x^4} + \&c \right\}$$

$$\frac{1}{2n^2} + \frac{1}{1^2+n+n^2} + \frac{1}{2^2+2n+n^2} + \frac{1}{3^2+3n+n^2} + \&c$$

$$+ 2n\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+n^2+n^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4+2^2n^2+n^4} + \&c \right\}$$

$$= \frac{1}{2\pi n^3\sqrt{3}} + \frac{2\pi}{3n\sqrt{3}} - \frac{2\pi}{n\sqrt{3}} \cdot \frac{1}{e^{2\pi n\sqrt{3}} - 2e^{\pi n\sqrt{3}}\cos \pi n + 1}$$

$$\frac{1}{6n^2} + \frac{1}{1^2+3n+3n^2} + \frac{1}{2^2+6n+3n^2} + \frac{1}{3^2+9n+3n^2} + \&c.$$

$$+ 6n\left\{ \frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4-3n^2+9n^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4-2^2\cdot 3n^2+9n^4} + \&c \right\}$$

$$= \frac{1}{6\pi n^3\sqrt{3}} + \frac{\pi}{3n\sqrt{3}} - \frac{2\pi}{n\sqrt{3}} \cdot \frac{1}{e^{2\pi n\sqrt{3}} - 2e^{\pi n\sqrt{3}}\cos 3\pi n + 1}$$

Cor 5. $\frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n+1} \cdot \frac{11^n}{11^n+1} \cdots \cdots \frac{p^n}{p^n - \sin\frac{\pi p}{2}} \cdots$ ad inf.

where p is a prime number

$= \frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \text{\&c}$

8. $(1+a_2)(1+a_3)(1+a_5)(1+a_7)(1+a_{11})$ &c where 2, 3, 5, 7 &c are prime numbers

$= 1 + a_2 + a_3 + a_5 + a_2 a_3 + a_7 + a_2 a_5 + a_{11} + a_{13} + \text{\&c}$

where the suffixes are natural numbers resolved into prime factors no two of which are alike.

Cor 1. $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{\&c} = \frac{S_n}{S_{2n}}$

Cor 2. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \frac{1}{19^n}$

$+ \frac{1}{23^n} + \frac{1}{29^n} + \frac{1}{30^n} + \frac{1}{31^n} + \frac{1}{37^n} + \text{\&c} = \frac{(S_n)^2 - S_{2n}}{2 S_n S_{2n}}$

where 2, 3, 5, 7, &c are natural numbers containing an odd number of prime factors no two of which are alike.

Cor 3. $\frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{12^n} + \text{\&c} = \frac{S_n(S_{2n} - 1)}{S_{2n}}$

where 4, 8, 9, 12 &c are composite numbers containing at least two equal prime numbers.

Ex 1. Show that the sum of the reciprocals of all prime numbers is infinite.

Sol. From 7 Cor 1 we have

$\frac{2}{2-1} \cdot \frac{3}{3-1} \cdot \frac{5}{5-1}$ &c $= 1 + \frac{1}{2} + \frac{1}{3} + \text{\&c}$

$\therefore \log_e \frac{2}{2-1} + \log_e \frac{3}{3-1} + \log_e \frac{5}{5-1} + \text{\&c} = \log_e\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{\&c}\right)$

i.e. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{\&c}$ + a finite quantity [illegible]

the sum of the reciprocals of all prime nos. ...

2. Shew that when $n = 0$, $\log_e n + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \&c$ is a finite quantity where 2, 3, 5, 7 &c are prime numbers.

Sol. From VII we have

$$\frac{2^{n+1}}{2^{n+1}-1} \cdot \frac{3^{n+1}}{3^{n+1}-1} \cdot \frac{5^{n+1}}{5^{n+1}-1} \&c = \frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \&c$$

$$\log_e \frac{2^{n+1}}{2^{n+1}-1} + \log_e \frac{3^{n+1}}{3^{n+1}-1} + \log_e \frac{5^{n+1}}{5^{n+1}-1} + \&c = \log_e \left(\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \&c\right)$$

$= -\log_e n$ when $n = 0$

$\therefore \log_e n + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \&c$ is a finite quantity when $n = 0$. + . 312

3. If 2, 3, 5, 7, 11 are all prime numbers shew that

i. $\frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \&c = \frac{5}{2}$

ii. $\frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \&c = \frac{7}{6}$

4. If 2, 3, 5, 7, 8 &c are natural numbers ~~are natural~~ containing an odd no. of prime factors ~~numbers~~ shew that

i. $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c = \frac{\pi^2}{20}$

ii. $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \&c = \frac{\pi^4}{1260}$

5. $\frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2} + \&c$ is a convergent series 2, 3, 5, 7 &c being prime nos.

$$\frac{\frac{\log_e 1}{1^{2}} + \frac{\log_e 2}{2^{2}} + \frac{\log_e 3}{3^{2}} + \frac{\log_e 4}{4^{2}} + \&c}{\frac{1}{1^{2}} + \frac{1}{2^{2}} + \frac{1}{3^{2}} + \frac{1}{4^{2}} + \&c} = \frac{\log_e 2}{2^2-1} + \frac{\log_e 3}{3^2-1} + \frac{\log_e 5}{5^2-1} + \&c$$

where 2, 3, 5, 7, 11 are prime nos.

$$f(1) + f(2) + f(3) + \cdots + f(h)$$

$$= \{f(1) + f(2) + f(3) + \cdots + f(n)\}$$

$$- \{f(1+h) + f(2+h) + \cdots + f(n+h)\}$$

$$+ h f(n) + \frac{\Sigma h}{\underline{|1}} f'(n) + \frac{\Sigma h^2}{\underline{|2}} f''(n) + \&c$$

N.B. When n becomes infinite we may neglect $f'(n)$ and the terms succeeding $f^2(n)$ if it is 0.

e.g. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{h}$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$- \left(\frac{1}{1+h} + \frac{1}{2+h} + \frac{1}{3+h} + \cdots + \frac{1}{n+h}\right) \text{ when } n = \infty.$$

$$= C_0 + \log_e n - \left(\frac{1}{1+h} + \frac{1}{2+h} + \frac{1}{3+h} + \cdots \frac{1}{n+h}\right) \text{ when } n = \infty$$

$$\underline{|h} = \frac{n^h}{\left(1+\frac{h}{1}\right)\left(1+\frac{h}{2}\right)\left(1+\frac{h}{3}\right)\cdots\left(1+\frac{h}{n}\right)} \text{ when } n = \infty.$$

$$f(1) + f(2) + f(3) + \cdots + f(h)$$

$$= x f(1) - x^{1+h} f(1+h) + x^2 f(2) - x^{2+h} f(2+h) + \&c$$

when x becomes unity.

CHAPTER VIII

1. $f(1) + f(2) + f(3) + f(4) + \dots\dots\dots + f(x) = \phi(x)$.

$$\phi(x) = C + \int f(x)\,dx + \tfrac{1}{2} f(x) + \frac{B_2}{\lfloor 2} f'(x) - \frac{B_4}{\lfloor 4} f'''(x) + \frac{B_6}{\lfloor 6} f^{v}(x) - \frac{B_8}{\lfloor 8} f^{vii}(x) + \&c$$

Sol. $\phi(x) - \phi(x-1) = f(x)$; apply VI 1.

N.B. 1. By giving any value to x, C can be found. R.H.S is not a terminating series except in some special cases. Consequently no constant can be found in $\tfrac{1}{2} f(x) + \frac{B_2}{\lfloor 2} f'(x) - \frac{B_4}{\lfloor 4} f'''(x) + \&c$ except in those special cases. If R.H.S be a terminating series it must be some integral function of x, in this case there is no possibility of a constant (according to the ordinary sense) in $\phi(x)$; for, $\phi(1) = f(1) + \phi(0)$. But $\phi(1) = f(1)$ ∴ $\phi(0)$ is always 0 whether $\phi(x)$ is rational or irrational. ∴ When $\phi(x)$ is a rational integral function of (x) it must be divisible by x since $\phi(0) = 0$ and consequently no constant but 0 can exist. Therefore let us try to give some other meaning for the constant of a series.

The constant of a series is the constant obtained by completing the remaining part by faithfully adhering to the above rule for summation of series. The constant of a series has some mysterious connection with the given infinite series and it is like the centre of gravity of a body. Mysterious because we may substitute it for the divergent infinite series. Now the constant of the series $1+1+1+1+\&c$ is $-\tfrac{1}{2}$ for the sum of x terms $= x = C + \int 1\,dx + \tfrac{1}{2}$ ∴ $C = -\tfrac{1}{2}$.

$\sqrt{21}$ $\frac{1}{2}\left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2 \left(\sqrt{\frac{5+\sqrt{7}}{4}} - \sqrt{\frac{1+\sqrt{7}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{7}}{4}} - \sqrt{\frac{\sqrt{7}-1}{4}}\right)^4 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3$

$\sqrt{33}$ $\frac{1}{2}(2-\sqrt{3})^3 \left(\sqrt{\frac{7+3\sqrt{3}}{4}} - \sqrt{\frac{3+3\sqrt{3}}{4}}\right)^4 \left(\sqrt{\frac{5+\sqrt{3}}{4}} - \sqrt{\frac{1+\sqrt{3}}{4}}\right)^4 \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)$

$\sqrt{45}$ $\frac{1}{2}(\sqrt{5}-2)^3 \left(\sqrt{\frac{7+3\sqrt{5}}{4}} - \sqrt{\frac{3+3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{5}}{2}} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)^4 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4$

Another way of finding the constant is as follows :— Let us take the series $1+2+3+4+5+$ &c. Let C be its constant. Then $C = 1+2+3+4+$ &c

$$\therefore 4C = \quad 4 \quad + 8 \quad + \text{\&c}$$

$$\therefore -3C = 1-2+3-4+\text{\&c} = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$$\therefore C = -\frac{1}{12}.$$

N.B. 2. For finding the sum to a fractional number of terms assume the sum to be true always and [illegible] where there is anything difficult in finding $\phi(h)$ where h is small, take n any integer you choose, find $\phi(n+h)$ and then subtract $\{f(1+h)+f(2+h)+\cdots+f(n+h)\}$ from the result.

The sum to a negative number of terms is the sum with the sign changed, calculated backward from the term previous to the first to the given no. of terms with positive sign instead of negative.

Cor. $\phi(x) = \sum_{n=0}^{n=\infty} \frac{B_n}{|\underline{n}} f^{n-1}(x) \cos\frac{\pi n}{2}$

Sol. Let $\frac{B_n}{|\underline{n}}\psi(n)$ be the coeff^t. of $f^{n-1}(x)$, then we see

$\psi(0)=1$, $\psi(2)=-1$, $\psi(4)=1$, $\psi(6)=-1$ &c

$\psi(3)=0$, $\psi(5)=0$, $\psi(7)=0$ &c. $\frac{B_1}{|\underline{1}}\psi(1)=\frac{1}{2}$ But $B_1 = \infty$

$\therefore \psi(1)=0$. [illegible] by VII of cor 2 we have $\pi(n-1)B_n = 1$ when $n=1$. $\therefore \frac{B_n\psi(n)}{|\underline{n}} = \frac{\pi(n-1)B_n}{|\underline{n}} \cdot \frac{\psi(n)}{\pi(n-1)} = \frac{1}{2}$ when $n=1$, [illegible] $\frac{\psi(n)}{\pi(n-1)} = \frac{1}{2}$ when $n=1$. $\therefore \psi(n) = -\cos\frac{\pi n}{2}$

$$\therefore \phi(x) + \sum_{n=0}^{n=\infty}\left\{\frac{B_n}{|\underline{n}} f^{n-1}(x)\cos\frac{\pi n}{2}\right\} = 0.$$

$2 \quad (\sqrt{2}-1)^4$

$2\sqrt{3} \quad (\sqrt{3}-\sqrt{2})^4(\sqrt{2}-1)^4$

$2\sqrt{7} \quad (\sqrt{2}-1)^8\,(2\sqrt{2}-\sqrt{7})^4$

$2\sqrt{15} \quad (\sqrt{10}-3)^4(\sqrt{6}-\sqrt{5})^4(\sqrt{3}-\sqrt{2})^4(\sqrt{2}-1)^4.$

$2\sqrt{2} \quad \left(\sqrt{3+2\sqrt{2}}-\sqrt{2+2\sqrt{2}}\right)^4$

$2\sqrt{6} \quad \left(\sqrt{6+3\sqrt{3}}-\sqrt{5+3\sqrt{3}}\right)^4\left(\sqrt{2+\sqrt{3}}-\sqrt{1+\sqrt{3}}\right)^4$

$2\sqrt{10} \quad \left(2\sqrt{2}+\sqrt{5}-2\sqrt{3+\sqrt{10}}\right)^4\left(\sqrt{2}+\sqrt{5}-\sqrt{6+2\sqrt{10}}\right)^4.$

If $e^{-\pi\sqrt{2p}} = F\,(\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2$

then $e^{-2\pi\sqrt{2p}} = F\left\{\frac{\sqrt{n}+1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{(\sqrt{n}+1)(\sqrt{n}+\sqrt{n+1})}\right\}^4$

$\times\left\{\frac{\sqrt{n}-1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{(\sqrt{n}-1)(\sqrt{n}+\sqrt{n+1})}\right\}^4.$

If $e^{-\pi\sqrt{p}} = F\,\frac{1-\sqrt{1-\frac{1}{n^2}}}{2}$

then $e^{-2\pi\sqrt{p}} = F\,(\sqrt{n+1}-\sqrt{n})^4(\sqrt{n}-\sqrt{n-1})^4.$

& $e^{-4\pi\sqrt{p}} = F\left\{(\sqrt{n+1}+\sqrt{n})(\sqrt{2n}+1)-\sqrt{\cdots}\right\}^4$

$\times\left\{(\sqrt{n+1}+\sqrt{n})(\sqrt{2n}-1)-\sqrt{\cdots}\right\}^4$

$= F\,(\sqrt{n+1}+\sqrt{n})^8\left\{\sqrt{2n}+1-\sqrt{2\sqrt{n}(\sqrt{n+1}+\sqrt{2})}\right\}^4$

$\times\left\{\sqrt{2n}-1-\sqrt{2\sqrt{n}(\sqrt{n+1}-\sqrt{2})}\right\}^4.$

2. Def. A series is said to be corrected when its constant is subtracted from it.

Theorem – The differential coeff.ᵗ of a series is a corrected series.

i.e. $\frac{d\{\phi(1)+\phi(2)+\phi(3)+\dots+\phi(x)\}}{dx} = \phi'(1)+\phi'(2)+\dots+\phi'(x) - c'$

where c' is the constant of $\phi'(1)+\phi'(2)+\phi'(3)+\dots+\phi'(x)$

Sol. In the differential coeff.ᵗ of $\phi(1)+\phi(2)+\dots+\phi(x)$ there can't be any constant. Therefore it should be corrected.

V.B. If $f(1)+f(2)+\dots+f(x)+\&c$ be a convergent series then its constant is the sum of the series itself.

Ex. $\frac{d\,\Sigma\frac{1}{x}}{dx} = \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$

Sol. $\frac{d\,\Sigma\frac{1}{x}}{dx} = -\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\dots-\frac{1}{x^2}-c$

$+\&c$ by VIII & V.B

$= \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}$

2. If S_1 be the constant of $\Sigma\frac{1}{x}$, then $\frac{d\,\lfloor x}{dx} = \lfloor x\,(\Sigma\frac{1}{x} - S_1)$

Sol. $\frac{d\,\lfloor x}{dx} = \lfloor x\,\frac{d\log_e\lfloor x}{dx} = \lfloor x\,(\Sigma\frac{1}{x}-S_1)$

3. $\int_0^x \Sigma\frac{1}{x}\,dx = \log_e\lfloor x + S_1 x$

4. $\int_0^x (1^{11}+2^{11}+3^{11}+\dots x^{11})\,dx = \frac{1}{12}(1^{12}+2^{12}+\dots+x^{12}) - \frac{x}{12}$

5. $\frac{d(1^{10}+2^{10}+3^{10}+\dots+x^{10})}{dx} = 10(1^9+2^9+\dots+x^9) + \frac{1}{132}$

6. $\int_0^x \{\sqrt{1}+\sqrt{2}+\sqrt{3}+\dots+\sqrt{x}\}\,dx = \frac{2}{3}(1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\dots+x\sqrt{x})$

$-\frac{3}{2\pi}\left(\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\&c\right)$

If $S_n = \frac{B_n}{2n} + (-1)^{\frac{n}{2}} \left\{ \frac{1^{n-1}}{x-1} + \frac{2^{n-1}}{x^2-1} + \frac{3^{n-1}}{x^3-1} + \&c \right\}$

then $S_{n+2} \frac{(n+2)(n+3)}{2} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 2} S_4 S_{n-2} (n-2)(n-13)$

$+ \frac{n \ldots (n-5)}{\lfloor 4} S_6 S_{n-4} (n-7)(n-18)$

~~n is even & gr~~

$+ \frac{n \ldots (n-7)}{\lfloor 6} S_8 S_{n-6} \{(n-12)(n-23) - 5.1.6\}$

$+ \frac{n \ldots (n-9)}{\lfloor 8} S_{10} S_{n-8} \{(n-17)(n-28) - 5.2.7\}$

$+ \frac{n \ldots (n-11)}{\lfloor 10} S_{12} S_{n-10} \{(n-22)(n-33) - 5.3.8\} + \&c$

If the last term be a perfect sq then take ½ the term only.

n is any even no. greater than 6. $S_8 = 120 S_4^2$

3. If $f^n(x)$ stands for the nth derivative of $f(x)$ and c_n be the constant of $\{f^n(1) + f^n(2) + \dots + f^n(x)\}$, then

$\phi(x) = -c_1 x - c_2 \frac{x^2}{\lfloor 2} - c_3 \frac{x^3}{\lfloor 3} - c_4 \frac{x^4}{\lfloor 4}$ &c

Sol. We know $\phi(x) = \phi(0) + \frac{x}{\lfloor 1}\phi'(0) + \frac{x^2}{\lfloor 2}\phi''(0) +$ &c

From VIII we have $\phi(0) = 0$, $\phi'(0) = -c_1$, $\phi''(0) = -c_2$ &c

Ex. 1. Shew that $\log_e \lfloor x = -S_1 x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 -$ &c

where S_n is the constant of $(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} +$ &c$)$

2. $x\,\frac{1}{x} = S_2 x - S_3 x^2 + S_4 x^3 - S_5 x^4 +$ &c, where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots$

N.B. This is very useful in finding $\phi(x)$ for fractional values of x

4. If $c_{\frac{1}{n}}$ be the constant of $f(\frac{1}{n}) + f(\frac{2}{n}) + f(\frac{3}{n}) + \dots + f(\frac{x}{n})$

Then $\phi(\frac{x}{n}) + \phi(\frac{x-1}{n}) + \phi(\frac{x-2}{n}) + \dots + \phi(\frac{x-n+1}{n}) - nc$

$= f(\frac{1}{n}) + f(\frac{2}{n}) + f(\frac{3}{n}) + \dots + f(\frac{x}{n}) - c_{\frac{1}{n}}$

Sol. Let $\psi(x) = \phi(\frac{x}{n}) + \phi(\frac{x-1}{n}) + \dots + \phi(\frac{x-n+1}{n})$ then

$\psi(x) - \psi(x-1) = \phi(\frac{x}{n}) - \phi(\frac{x-n}{n}) = f(\frac{x}{n})$

$\therefore \psi(x)$ and $f(\frac{x}{n}) + f(\frac{x-1}{n}) + \dots f(\frac{1}{n})$ differ only by some constant; hence if these be corrected they must be equal. $\psi(x)$ contains n terms each of which is of the form $\phi(x)$ whose constant is c. ∴ The constant of $\psi(x) = nc$ and the constant of $f(\frac{1}{n}) + f(\frac{2}{n}) + f(\frac{3}{n}) + \dots + f(\frac{x}{n})$ is $c_{\frac{1}{n}}$ by our supposition.

Cor. $\phi(-\frac{1}{n}) + \phi(-\frac{2}{n}) + \phi(-\frac{3}{n}) + \dots + \phi(-\frac{n-1}{n}) = nc - c_{\frac{1}{n}}$

Sol. Put $x = 0$ in the above theorem

Ex. 1 Shew that $\phi(-\frac{1}{2}) = 2c - c_{\frac{1}{2}}$

2. $c = \dots$

$\sqrt[16]{256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} = P$ & $\sqrt[16]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1, 3, 5, 15 : — $Q^3 + \frac{1}{Q^3} - \sqrt{2}\left(\frac{1}{P} + P\right) = 0.$

1, 3, 7, 21 : — $\left(Q^{16} + \frac{1}{Q^{16}}\right) - 5\left(Q^{12} + \frac{1}{Q^{12}}\right) + 5\left(Q^{8} + \frac{1}{Q^{8}}\right) + 6\left(Q^{4} + \frac{1}{Q^{4}}\right)$
$- \left(\frac{8}{P^6} - 6 + 8P^6\right) = 0$

1, 5, 7, 35 : — $Q^4 + \frac{1}{Q^4} - \left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0$

~~1, 5, 11, 55~~ — { 3, 1, 5, 15 : — $\left(Q^4 + \frac{1}{Q^4}\right) - 2\left(P^2 + \frac{1}{P^2}\right) + 3 = 0.$
5, 1, 3, 15 : — $\left(Q^6 + \frac{1}{Q^6}\right) - 4\left(P^4 + \frac{1}{P^4}\right) + 10\left(P^2 + \frac{1}{P^2} - 1\right) = 0$

7, 1, 3, 21 $\left(Q^8 + \frac{1}{Q^8}\right) + 7\left(Q^6 + \frac{1}{Q^6}\right) + 14\left(Q^4 + \frac{1}{Q^4}\right) + 21\left(Q^2 + \frac{1}{Q^2}\right) + 42$
$- 8\left(P^6 + \frac{1}{P^6}\right) = 0$

5, 1, 7, 35. $\left(Q^6 + \frac{1}{Q^6}\right) + 5\sqrt{2}\left(Q^3 + \frac{1}{Q^3}\right)\left(P + \frac{1}{P}\right) + 10$
$- 4\left(P^4 + \frac{1}{P^4}\right) = 0.$

5, 1, 11, 55 $\left(Q^6 + \frac{1}{Q^6}\right) - 5\left(Q^4 + \frac{1}{Q^4}\right) + 10\left(Q^2 + \frac{1}{Q^2}\right)\left(\frac{1}{P^2} - 1 + P^2\right)$
$- \left(\frac{4}{P^4} - \frac{10}{P^2} + 25 - 10P^2 + 4P^4\right) = 0.$

3, 1, 11, 33 $Q^4 + \frac{1}{Q^4} + 3\left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0$

$\sqrt[8]{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}} = P$ & $\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1, 5, 7, 35 $\left(P^4 + \frac{1}{P^4}\right) - \left(Q^6 + \frac{1}{Q^6}\right) + 5\left(Q^4 + \frac{1}{Q^4}\right) - 10\left(Q^2 + \frac{1}{Q^2}\right) + 15 = 0$

3. [illegible] $+ \phi(-\frac{1}{3}) =$ [illegible]

4. [illegible] $+ \phi(-\frac{1}{4}) =$ [illegible]

5. [illegible] $+ \phi(-\frac{1}{5}) = c + \frac{c_1}{2} + \frac{c_3}{3} \ldots$ [illegible]

6. [illegible] $= c + \int$ [illegible] dx [illegible] $+$ [illegible] &c

$= \sum$ [illegible] $\cos$ [illegible]

5. $S(a_1 + a_2 + a_3 + \&c)$ means that the series is a convergent one and its sum to infinity is req^d.

6. $D(a_1 + a_2 + a_3 + \&c)$ means that the series is a pure divergent series and its constant is req^d.

7. $G(a_1 + a_2 + a_3 + \&c)$ means that the series is an oscillating series (convergent or divergent) or that the series is a pure divergent series whose sum to n terms can [illegible] and consequently its constant [illegible] and that in both cases the value of the generating function is required.

8. Hereafter the series will only be given [illegible] and from the nature of the series we [illegible] whether S or G [illegible]. Moreover if a series [illegible] a generating function we must select S or G from the nature of the series.

9. [illegible] series is only true when the series is [illegible] from a regular series. For example the value of $1 - 1 + 1 - 1 + \&c = \frac{1}{2}$ only when it is deduced from a series of the form [illegible] + &c [illegible]

$a_1 - a_2 + a_3 - a_4 + \&c$

$$= \frac{a_1}{2} + \frac{a_1 - a_2}{4} + \frac{a_1 - 2a_2 + a_3}{8} + \&c$$

$a_1 - a_2 + a_3 - a_4 + \&c$

$$= xa_1 - x^2a_2 + x^3a_3 - x^4a_4 + \&c$$

$$= x\frac{a_1}{2} + x^2\frac{a_1 - a_2}{4} + x^3\frac{a_1 - 2a_2 + a_3}{8} + \&c$$

} when x approaches unity.

$$\sqrt[48]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}} = P \qquad \sqrt[48]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$$

1, 13, 5, 65: $-\left(Q^6 + \frac{1}{Q^6}\right) - 5\left(Q^2 + \frac{1}{Q^2}\right)^2\left(P + \frac{1}{P}\right)^2$
$- \left(\frac{4}{P^4} + P^4\right) = 0.$

1, 13, 3, 39 $-\left(Q^4 + \frac{1}{Q^4}\right) - 3\left(Q^2 + \frac{1}{Q^2}\right) + 3 - \left(P^2 + \frac{1}{P^2}\right) = 0$

~~1, 3, 11, 33~~ —

we [illegible] the [illegible] series $1 - 1 + 1 - 1 + \&c$ when n becomes [illegible]

yet its value is [illegible] $\frac{1}{2}$ in this case

V. $a_1 - a_2 + a_3 - a_4 + \&c$ is not equal to the series

$(a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \&c$ or to the series

$a - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \&c$. For example

$1 - 2 + 3 - 4 + 5 - 6 + \&c$ is not equal to the series

$(1-2) + (3-4) + (5-6) + \&c$ or the series $1 - (2-3) - (4-5) - \&c$.

But $a_1 - a_2 + a_3 - a_4 + \&c = a_1 - (a_2 - a_3 + a_4 - a_5 + \&c)$

VI. $(a_1 - a_2 + a_3 - a_4 + \&c) + (b_1 - b_2 + b_3 - b_4 + \&c)$

$= (a_1 + b_1) - (a_2 + b_2) + (a_3 + b_3) - (a_4 + b_4) + \&c$

Cor. $(a_1 - a_2 + a_3 - a_4 + \&c) + (b_1 - b_2 + b_3 - b_4 + \&c)$

$= a_1 + (b_1 - a_2) - (b_2 - a_3) + (b_3 - a_4) - (b_4 - a_5) + \&c$

Sol. [illegible] we know

$a_1 - a_2 + a_3 - a_4 + \&c = a_1 - (a_2 - a_3 + a_4 - \&c)$

$\therefore (a_1 - a_2 + a_3 - a_4 + \&c) + (b_1 - b_2 + b_3 - b_4 + \&c)$

$= a_1 + (b_1 - b_2 + b_3 - b_4 + \&c) + (-a_2 + a_3 - a_4 + \&c)$

$= a_1 + (b_1 - a_2) - (b_2 - a_3) + (b_3 - a_4) - \&c$

Ex. 1. $a_1 - a_2 + a_3 - a_4 + \&c = \frac{a_1}{2} + \frac{1}{2}\{(a_1 - a_2) - (a_2 - a_3) + \&c\}$

2. $a_1 - a_2 + a_3 - a_4 + \&c = \frac{3a_1 - a_2}{4} + \frac{1}{4}\{(a_1 - 2a_2 + a_3) - (a_2 - 2a_3 + a_4) + (a_3 - 2a_4 + a_5) - \&c\}$

3. $a_1 - a_2 + a_3 - a_4 + \&c = \frac{7a_1 - 4a_2 + a_3}{8}$ [illegible]

$+ \frac{1}{8}\{(a_1 - $ [illegible] $(a_2 - 3a_3 + 3a_4 - a_5)$ [illegible]

$$\sqrt[24]{16\alpha\beta(1-\alpha)(1-\beta)} = P \text{ and } \sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = Q$$

3. $Q^6 + \frac{1}{Q^6} - 2\left(\frac{1}{P^3} - P^3\right) = 0$

5. $Q^3 + \frac{1}{Q^3} - 2\left(\frac{1}{P^2} - P^2\right) = 0$

7. $Q^4 + \frac{1}{Q^4} - 2\left(\frac{\sqrt{2}}{P^3} - \frac{7}{2} + \sqrt{2}P^3\right) = 0$

11. $Q^6 + \frac{1}{Q^6} - 2\sqrt{2}\left(\frac{2}{P^5} - \frac{11}{P^3} + \frac{22}{P} - 22P + 11P^3 - 2P^5\right) = 0$

13. $Q^7 + \frac{1}{Q^7} + 13\left(Q^5 + \frac{1}{Q^5}\right) + 52\left(Q^3 + \frac{1}{Q^3}\right) + 78\left(Q + \frac{1}{Q}\right)$
$- 8\left(\frac{1}{P^6} - P^6\right) = 0.$

17. $Q^9 + \frac{1}{Q^9} - 34\left(Q^6 + \frac{1}{Q^6}\right) + 17\left(Q^3 + \frac{1}{Q^3}\right)\left(\frac{4}{P^4} + 7 + 4P^4\right)$
$- \left(\frac{16}{P^8} - \frac{136}{P^4} - 340 - 136P^4 + 16P^8\right) = 0$

19. $Q^{10} + \frac{1}{Q^{10}} + 114\left(Q^6 + \frac{1}{Q^6}\right) - 190\sqrt{2}\left(Q^4 + \frac{1}{Q^4}\right)\left(\frac{1}{P^3} - P^3\right)$
$+ 19\left(Q^2 + \frac{1}{Q^2}\right)\left(\frac{8}{P^6} - 5 + 8P^6\right)$
$- 4\sqrt{2}\left(\frac{4}{P^9} + \frac{19}{P^3} - 19P^3 - 4P^9\right) = 0.$

These are true even for even functions though the signs are changed in most cases.

vii. If $\frac{a_1}{a_2}$ [illegible] $\frac{a_2}{a_3}$ [illegible] then $\frac{a_1^2}{a_1+a_2}$ and a_1 $\frac{a_2^2}{a_2+a_3}$

$a_1 - a_2 + a_3 - \&c$ lies between [illegible]

Ex. $1 - 2 + 3 - 4 + \&c$ lies between $\frac{1}{3}$ and $\frac{1}{5}$ and its value

(i) $\frac{1}{4}$ [illegible] $+ [illegible] - [illegible] + \&c$ lies between $\frac{1}{2}$ and $\frac{1}{3}$; its value is $\frac{2}{5}$ [illegible].

But $2 - 2\frac{1}{2} + 3\frac{1}{3} - 4\frac{1}{4} + 5\frac{1}{5} - \&c$ cannot lie between

$\frac{2^2}{2+2\frac{1}{2}}$ and $2 - \frac{(2\frac{1}{2})^2}{2\frac{1}{2}+3\frac{1}{3}}$ as $\frac{2\frac{1}{2}}{3\frac{1}{3}}$ is not lying between

$\frac{2}{2\frac{1}{2}}$ and $\frac{3\frac{1}{3}}{4\frac{1}{4}}$

6. $\phi_1(x) + \phi_2(x) + \phi_3(x) + \&c$ can be expanded in as[illegible] [illegible] powers of x say $A_0 + A_1 x + [illegible] x^2 + A_3 x^3 + \&c$ where each of A_0, A_1, A_2 &c is a series.

Case I When A_n is a convergent series

(1) If $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c$ be a rapidly convergent series what is required is got.

(2) But if it is a slowly convergent or an oscillating series, convergent [illegible] (at least for [illegible] values of x)

(a) change x into a suitable [illegible] of y so that the new series [illegible] in ascending powers of y may be [illegible] [illegible] [illegible]

$\sqrt[8]{\alpha\beta} = P$ $\sqrt[8]{\frac{\beta}{\alpha}} = Q$

3. $\frac{1}{Q^2} - Q^2 - 2\left(\frac{1}{P} - P\right) = 0$

5. $\frac{1}{Q^3} - Q^3 + 5\left(\frac{1}{Q} - Q\right) - 4\left(\frac{1}{P^2} - P^2\right) = 0$

7. $\frac{1}{Q^4} + Q^4 - 8\left(\frac{1}{P^3} + P^3\right) + 28\left(\frac{1}{P^2} + P^2\right) - 56\left(\frac{1}{P} + P\right) + 70 = 0$

[illegible] $\frac{12}{x^3} - \frac{12}{x^4} + \text{\&c} = \frac{1}{x+1} - \frac{1^2}{x+1+} \frac{2^2}{x+1+}$ &c
$\frac{x}{3+} \frac{x}{5+} \frac{x}{7+\text{\&c}}$

(c) or transform it into another series by applying III [illegible]

Eg $\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{15}{x^4} + \frac{52}{x^5} -$ &c

$= \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} + \frac{1}{(x+1)(x+2)(x+3)} -$ &c

(d) or take the reciprocal of the series and try to make it a rapidly convergent series in any way

Case II When it is an oscillating (convergent or divergent) or a pure divergent series.

(1) Let C_n be the constant or the value of its generating function according as it is purely divergent or not. Then the given series $= F(x) + C_0 + C_1 x + C_2 x^2 + C_3 x^3 +$ &c where $F(x)$ is a simple function of x

(2) But if unfortunately $C_0 + C_1 x + C_2 x^2 + C_3 x^3 +$ &c be a divergent series, find some function of n (say P_n) such that the value of $P_0 + P_1 x + P_2 x^2 +$ &c may easily be found and $C_n - P_n$ may rapidly diminish as n increases. Then the given series $= F(x) + (C_0 - P_0) + (C_1 - P_1) x + (C_2 - P_2) x^2 + (C_3 - P_3) x^3 +$ &c

Eg. 1. [illegible] &c [illegible] $\frac{1}{2}(1 -$ [illegible] $) - \frac{1}{2x}(1 -$ [illegible]

$=$ [illegible] $\frac{1}{2x}$ + &c

2. [illegible] x^2 + &c [illegible] $\frac{1}{2}$ ([illegible] &c)

[illegible] &c) $- \frac{1}{24}$ ([illegible] &c) [illegible]

$= \frac{1}{2}$ [illegible] $\frac{\pi \cot \pi x}{2x}$

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2n} dx = \frac{\pi n}{(2n-1)(n+1)}\left(1+\frac{1}{2}+\dots\frac{1}{n}\right).$$

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2n+1} dx = 2\pi \frac{1^2.3^2\dots(2n-1)^2}{2^2.4^2\dots 2n^2}\left(1+\frac{1}{3}+\dots\frac{1}{2n-1}\right),$$

$$1+\frac{\frac13.\frac23}{1.\frac12}x+\frac{\frac13.\frac23.\frac43.\frac53}{1.2.\frac12.1\frac12}x^2+\&c$$

$$=\frac{\cos\left(\frac13\sin^{-1}\sqrt{x}\right)}{\sqrt{1-x}}.$$

$$1-\frac{\frac13.\frac23}{1.\frac12}x+\frac{\frac13.\frac23.\frac43.\frac53}{1.2.\frac12.1\frac12}x^2-\&c$$

$$=\frac{\sqrt[3]{\sqrt{1+x}+\sqrt{x}}+\sqrt[3]{\sqrt{1+x}-\sqrt{x}}}{2\sqrt{1+x}}.$$

$$1+\frac{\frac13.\frac23}{1.1\frac12}x+\frac{\frac13.\frac23.\frac43.\frac53}{1.2.1\frac12.2\frac12}x^2+\&c$$

$$=\frac{3}{\sqrt{x}}\sin\left(\frac13\sin^{-1}\sqrt{x}\right)$$

$$1-\frac{\frac13.\frac23}{1.1\frac12}x+\frac{\frac13.\frac23.\frac43.\frac53}{1.2.1\frac12.2\frac12}x^2-\&c$$

$$=\frac{3}{2\sqrt{x}}\left(\sqrt[3]{\sqrt{1+x}+\sqrt{x}}-\sqrt[3]{\sqrt{1+x}-\sqrt{x}}\right)$$

$$\cos\left(2n\sin^{-1}\sqrt{x}\right),\quad \frac{\sin\left(2n\sin^{-1}\sqrt{x}\right)}{2n\sqrt{x}},\quad \frac{\cos 2n\sin^{-1}\sqrt{x}}{\sqrt{1-x}}$$

$$\frac{\sin\left(2n\sin^{-1}\sqrt{x}\right)}{2n\sqrt{x(1-x)}}.$$

3. [illegible] $(1+1+1+\&c)$

$- x(\log 1 + \log_e 1 + \log_e 1 + \&c) + \&c = -\frac{1}{2} - x \log_e \sqrt{2\pi} - \&c$

$= \frac{1}{x_1} + 1 + [illegible] + \&c - \frac{1}{2} - x \log_e \sqrt{2\pi} - \&c$

$= \frac{1}{x_1} + \frac{1}{2} + x(1 - \log_e \sqrt{2\pi}) - \&c$

$= \frac{1}{x_1} + \frac{1}{2} + (1 - .91894)x - \&c = \frac{1}{x_1} + .5 + .0810\ldots x - \&c$

Cor. 1. $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \frac{x}{e^{4x}+1} + \&c$

$= \log_e 2 - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2 \frac{x^4(2^4-1)}{4\lfloor 4} +$

$(B_6)^2 \frac{x^6(2^6-1)}{6\lfloor 6} + \&c.$

Sol. By chapter III 5 cor. $\frac{x}{e^{nx}+1} = \frac{x}{2} - B_2 \frac{n\, x^2(2^2-1)}{\lfloor 2}$

$+ B_4 \frac{n^3 x^4(2^4-1)}{\lfloor 4} - B_6 \frac{n^5 x^6(2^6-1)}{\lfloor 6} + \&c$

$\therefore \frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$

$= \frac{x}{2}(1+1+1+\&c) - B_2 \frac{x^2(2^2-1)}{\lfloor 2}(1+2+3+\&c)$

$+ B_4 \frac{x^4(2^4-1)}{\lfloor 4}(1^3+2^3+3^3+\&c) - \&c$

$= \psi(x) - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2 \frac{x^4(2^4-1)}{4\lfloor 4} + \&c.$

Now it is req^d. to find ψx.

the given series $= \frac{x}{e^x-1} - \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} - \&c$

Here the term excepting x and higher powers of x is $\log_e 2$ $\therefore \psi(x) = \log_e 2$.

Cor 2. $\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \frac{x}{e^{4x}-1} + \&c$

$= S_1 - \log_e x + \frac{x}{4} - (B_2)^2 \frac{x^2}{2\lfloor 2} - (B_4)^2 \frac{x^4}{4\lfloor 4} - (B_6)^2 \frac{x^6}{6\lfloor 6} - \&c$

Sol. Proceeding as in the previous theorem we have

the series $= \psi(x) + S_1 + \frac{x}{4} - (B_2)^2 \frac{x^2}{2\lfloor 2} - (B_4)^2 \frac{x^4}{4\lfloor 4} - \&c$

$$1+6\left(\frac{1}{e^{y}+e^{-y}+1}+\frac{1}{e^{2y}+e^{-2y}+1}+\&c\right)$$

$$=1+\frac{1\cdot 2}{3^2}x+\&c=z$$

$$\frac{1^2}{e^{y}+e^{-y}+1}+\frac{2^2}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}z^3$$

$$\frac{1^4}{e^{y}+e^{-y}+1}+\frac{2^4}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}z^5$$

$$\frac{1^6}{e^{y}+e^{-y}+1}+\frac{2^6}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}\left(1+\frac{4x}{3}\right)z^7$$

$$\frac{1^8}{e^{y}+e^{-y}+1}+\frac{2^8}{e^{2y}+e^{-2y}+1}+\&c=\frac{x}{27}(1+8x)z^9$$

$$1+2e^{-\frac{\pi y}{\sin\pi h}}+2e^{-\frac{4\pi y}{\sin\pi h}}+\&c$$

$$=u\sqrt{1+\frac{h(1-h)}{4^2}x+\&c}$$

where $y=\dfrac{1+\frac{h(1-h)}{4^2}(1-x)+\&c}{1+\frac{h(1-h)}{4^2}x+\&c}$

& u can be expressed in radicals in terms of x & h.

$$\phi=\theta+3\left\{\frac{\sin 2\theta}{1+2\cosh y}+\frac{\sin 4\theta}{2(1+2\cosh 2y)}+\&c\right\}$$

$$\theta z=\int_0^{\phi}\left\{1+\frac{1\cdot 2}{3^2}\cdot\frac{2}{1}\cdot x\sin^2\phi+\frac{1\cdot 2\cdot 4\cdot 5}{3^2\cdot 6^2}\cdot\frac{2\cdot 4}{1\cdot 3}x^2\sin^4\phi+\&c\right\}d\phi$$

$$=\int_0^{\phi}\frac{\cos\left\{\frac{1}{3}\sin^{-1}(\sqrt{x}\sin\phi)\right\}}{\sqrt{1-x\sin^2\phi}}d\phi.$$

Hence we obtain $\frac{x}{e^{x}+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{4x}+1}$ + &c

$= \left\{\frac{x}{e^{x}-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \&c\right\} - \left\{\frac{2x}{e^{2x}-1} + \frac{2x}{e^{4x}-1} + \frac{2x}{e^{6x}-1} + \&c\right\}$

$\therefore \psi(x) - \psi(2x) = \log_e 2$. Hence $\psi(x)$ ... x

Ex 1. Show that the constant in the series

$\frac{(1)}{1} + \ldots + \ldots$ &c $= \cdot 5909100$

2. Show that $\frac{1}{e+1} + \frac{1}{e^2+1} + \frac{1}{e^3+1} + \frac{1}{e^4+1}$ + &c

$= \frac{3}{4} + \frac{1}{48}\log_e 2$ very nearly.

3. $\frac{1}{1+1\frac{1}{9}} + \frac{1}{1+(1\frac{1}{9})^2} + \frac{1}{1+(1\frac{1}{9})^3} + \frac{1}{1+(1\frac{1}{9})^4}$ + &c

$= 6.331009$

4. $\frac{1}{1\frac{1}{9}-1} + \frac{1}{(1\frac{1}{9})^2-1} + \frac{1}{(1\frac{1}{9})^3-1} + \frac{1}{(1\frac{1}{9})^4-1}$ + &c

$= 27$ nearly

7. i. $\frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^3-1} + \frac{1}{x^4-1} + \frac{1}{x^5-1}$ + &c

$= \frac{1}{x}\cdot\frac{x+1}{x-1} + \frac{1}{x^4}\cdot\frac{x^2+1}{x^2-1} + \frac{1}{x^9}\cdot\frac{x^3+1}{x^3-1} + \frac{1}{x^{16}}\cdot\frac{x^4+1}{x^4-1}$ + &c

ii. $\frac{1}{x-1} - \frac{1}{x^2-1} + \frac{1}{x^3-1} - \frac{1}{x^4-1}$ + &c or

$\frac{1}{x+1} + \frac{1}{x^2+1} + \frac{1}{x^3+1} + \frac{1}{x^4+1}$ + &c

$= \frac{1}{x}\cdot\frac{x+1}{x-1} - \frac{1}{x^4}\cdot\frac{x^2+1}{x^2-1} + \frac{1}{x^9}\cdot\frac{x^3+1}{x^3-1} - \frac{1}{x^{16}}\cdot\frac{x^4+1}{x^4-1}$ + &c

Sol. $\frac{1}{x-1} = \frac{1}{x} + \ldots$

$\frac{1}{x^2-1} = \frac{1}{x^2} + \frac{1}{x^2(x^2-1)}$

$\frac{1}{x^3-1} = \frac{1}{x^3} + \frac{1}{x^6} + \frac{1}{x^6(x^3-1)}$

$\ldots + \frac{1}{x^8} + \frac{1}{x^{12}} + \ldots$

&c &c &c &c &c &c &c

[illegible]

$$\phi^2(x) + \phi^2(x^{20}) = \left\{\phi(x^5) + 2x f(x^{15}, x^{25})\right\}^2 + \left\{\phi(x^5) + 2x^4 f(x^5, x^{45})\right\}^2.$$

~~$\phi^2(x) - \phi^2(x^5) = 4x f(-x^5) f(-x^{20})$~~

$$\left\{\phi(x^{25}) + 2x f(x^{15}, x^{35})\right\}^2 + \left\{\phi(x^{25}) + 2x^4 f(x^5, x^{45})\right\}^2 = \phi^2(x) - 2\phi^2(x^5) + 3\phi^2(x^{25}).$$

$$\phi^2(x) - \phi^2(x^5) = 4x\,\chi(x) f(-x^5) f(-x^{20})$$

$\frac{1}{x}\cdot\frac{x+1}{x-1} + \frac{1}{x^4}\cdot\frac{x^{2}+1}{x^{2}-1} + \frac{1}{x^8}\cdot\frac{x^4+1}{x^4-1} + \frac{1}{x^{16}}\cdot\frac{x^8+1}{x^8-1} + \&c$

and $\frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^4-1} + \frac{1}{x^8-1} + \&c =$

$\frac{1}{x^2}\cdot\frac{x^2+1}{x^2-1} + \frac{1}{x^4}\cdot\frac{x^4+1}{x^4-1} + \frac{1}{x^8}\cdot\frac{x^8+1}{x^8-1} + \frac{1}{x^{16}}\cdot\frac{x^8+1}{x^8-1} + \&c$

8. $\frac{r}{1-ax} + \frac{r^2}{1-ax^2} + \frac{r^3}{1-ax^3} + \frac{r^4}{1-ax^4} + \&c$ to n terms

$= \left\{\frac{arx}{1-ax} + \frac{(arx^2)^2}{1-ax^2} + \frac{(arx^3)^3}{1-ax^3} + \frac{(arx^4)^4}{1-ax^4} + \&c \text{ to } n \text{ terms}\right\}$

$+ \left\{\frac{r-r^{n+1}}{1-r} + a\frac{(rx)^2-(rx)^{n+1}}{1-rx} + a^2\frac{(rx^2)^3-(rx^2)^{n+1}}{1-rx^2} + \&c \text{ to } n \text{ terms}\right\}$

Sol. $\frac{r}{1-ax} = \frac{arx}{1-ax} + r$

$\frac{r^2}{1-ax^2} = \frac{(arx^2)^2}{1-ax^2} + r^2 + ar^2x^2$

$\frac{r^3}{1-ax^3} = \frac{(arx^3)^3}{1-ax^3} + r^3 + ar^3x^3 + a^2r^3x^6$

$\frac{r^4}{1-ax^4} = \frac{(arx^4)^4}{1-ax^4} + r^4 + ar^4x^4 + a^2r^4x^8 + a^3r^4x^{12}$

&c &c &c &c &c &c &

Adding up all these to n terms we can get the result.

or. $\frac{r}{1-ax} + \frac{r^2}{1-ax^2} + \frac{r^3}{1-ax^3} + \frac{r^4}{1-ax^4} + \&c$

$= \left\{\frac{arx}{1-ax} + \frac{(arx^2)^2}{1-ax^2} + \frac{(arx^3)^3}{1-ax^3} + \frac{(arx^4)^4}{1-ax^4} + \&c\right\}$

$+ \left\{\frac{r}{1-r} + \frac{a(rx)^2}{1-rx} + \frac{a^2(rx^2)^3}{1-rx^2} + \frac{a^3(rx^3)^4}{1-rx^3} + \&c\right\}$

Sol. [illegible] infinite in the above theorem.

or. $\frac{1}{x-1} - \frac{1}{x}$ [illegible]

$=$ [illegible]

$$2^n + 6^n + 12^n + 20^n + \&c$$

$$= A_n + \frac{n}{\lfloor 1} A_{n+1} + \frac{n(n-1)}{\lfloor 2} A_{n+2} + \&c$$

where $A_n = (1^n + 2^n + 3^n + \&c)(1 + \cos \pi n)$

$$\log 2 \left\{ \frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \&c \right\}$$

$$+ (\log 2)^2 \left\{ \frac{1}{2\log 2 \log 4} + \frac{1}{3\log 3 \log 6} + \frac{1}{4\log 4 \log 8} + \&c \right\} = 1$$

$$\int_0^\infty \frac{\cos 2nx}{\cosh \pi\sqrt{x} + \cos \pi\sqrt{x}}\, dx$$

$$= \frac{e^{-n}}{\cosh \frac{\pi}{2}} - \frac{3e^{-9n}}{\cosh \frac{3\pi}{2}} + \frac{5e^{-25n}}{\cosh \frac{5\pi}{2}} - \&c$$

If $\alpha\beta = \frac{\pi^3}{4}$, then

$$\frac{1}{\cosh \sqrt{\alpha} + \cos \sqrt{\alpha}} - \frac{1}{3} \cdot \frac{1}{\cosh \sqrt{3\alpha} + \cos \sqrt{3\alpha}} +$$

$$+ \frac{1}{\cosh \frac{\pi}{2} \cosh \beta} - \frac{1}{3} \cdot \frac{1}{\cosh \frac{3\pi}{2} \cosh 9\beta} + \&c$$

$$= \frac{\pi}{8}$$

CHAPTER IX

1. [illegible] $\varphi_n(x) = \frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$ [illegible] $+ \frac{n}{2}\frac{B_2}{2}x^{n-1} - B_4\frac{n(n-1)(n-2)}{4}x^{n-3}$ [illegible] $+ B_6$ [illegible] $\frac{\varphi(a-1)}{}$ [illegible] π.

Sol. The corresponding series is found by applying VI 1.
Let C_n be the constant. Since $\varphi_n(1) - \varphi_n(x)$ must be divisible by x. The coefft of x^{n-n} = $-\frac{[n}{[n-r[r+1}$ [illegible] $\cos\frac{\pi(n+1)}{2}$
∴ The term independent of x = $-\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$
∴ $C_n = \frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$

2. $(a+b)^n + (a+2b)^n + (a+3b)^n + \dots + (a+xb)^n = b^n\left\{\varphi_n\left(x+\frac{a}{b}\right) - \varphi_n\left(\frac{a}{b}\right)\right\}$

3. $1^n - 2^n + 3^n - 4^n + \&c = (2^{n+1}-1)\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$ [illegible]
Sol. $(1-2^{n+1})S_n = (1-2^{n+1})(1^n+2^n+3^n+\&c) = (1^n+2^n+3^n+\&c) - 2^{n+1}(1^n+2^n+3^n+4^n+\&c) = 1^n - 2^n + 3^n - 4^n + \&c$

Cor. $\varphi_n\left(-\frac{1}{2}\right) = \left(2 - \frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$

Sol. $\varphi_n\left(-\frac{1}{2}\right) = 1^n - \left(\frac{1}{2}\right)^n + 2^n - \left(1\frac{1}{2}\right)^n + 3^n - \left(2\frac{1}{2}\right)^n + \&c$
$= -\frac{1}{2^n}(1^n - 2^n + 3^n - \&c) = \left(2 - \frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$

4. $\frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2} = S_n = \frac{(2\pi)^n}{2[n}B_n$
[illegible] find B_n for negative values of n
Sol. $\frac{B_{1-n}}{1-n}\cos\frac{\pi(1-n)}{2}$ is the constant of $1^{-n}+2^{-n}+3^{-n}+\&c$
∴ $\frac{B_{1-n}}{1-n}\cos\frac{\pi(1-n)}{2}$ is that of $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c = S_n$
∴ $\frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2} = \frac{(2\pi)^n}{2[n}B_n$
Since $C_{-n} = S_n$, S_{-n} is [illegible]

Ex. 1. $B_{-2} = 2S_3$; $B_{-4} = -4S_5$; $B_{-6} = 6S_7$; $B_{-8} = -8S_9$ &c
2. $[-\frac{1}{2} = \sqrt{\pi}$ [illegible] $\frac{B_{\frac{3}{2}}}{\frac{3}{2}}\sin\frac{\pi}{4} = \frac{(2\pi)^{\frac{1}{2}}}{2[\frac{1}{2}}B_{\frac{1}{2}}$

If $f(x) = \Sigma \frac{P_n}{p_n - x}$ & $\phi(x) = \Sigma \frac{Q_n}{q_n - x}$

then $f(x)\phi(y) = \Sigma \left\{ \frac{P_n}{p_n - x} \phi\left(p_n \frac{y}{x}\right) + \frac{Q_n}{q_n - y} f\left(q_n \frac{x}{y}\right) \right\}$

If $\Sigma(a_n + a_{-n}) = a_0 + (a_1 + a_{-1}) + (a_2 + a_{-2}) + \&c$

then $\Sigma(a_n + a_{-n})\,\Sigma(b_n + b_{-n})$

$= \Sigma(a_n b_n + a_{-n} b_{-n}) + \{\Sigma(a_{n+1} b_{n-1} + a_{1-n} b_{-1-n})$

$+ \Sigma'(a_{-n-1} b_{-n+1} + a_{n-1} b_{n+1})\}$

$+ \{\Sigma(a_{n+2} b_{n-2} + a_{2-n} b_{-n-2}) + \Sigma(a_{-n-2} b_{-n+2} + a_{n-2} b_{n+2})\}$

$+ \&c$

$+ \{\Sigma(a_{1-n} b_n + a_{1+n} b_{-n}) + \Sigma(a_{-n} b_{n-1} + a_n b_{-n-1})\}$

$+ \{\Sigma(a_{2-n} b_{1+n} + a_{2+n} b_{1-n}) + \Sigma(a_{-1-n} b_{n-2} + a_{-1+n} b_{-n-2})\}$

$+ \&c$

$\pi^2 x^2 \frac{\cos\theta x \cosh\theta x}{\sin\pi x \sinh\pi x}$

$= 1 + 4\pi x^4 \left\{ \frac{\cos\theta \cosh\theta}{(1^4 - x^4)\sinh\pi} \right.$

$\left. - \frac{2\cos 2\theta \cosh 2\theta}{(2^4 - x^4)\sinh 2\pi} + \&c \right.$

or $\frac{B_{-\frac{1}{3}}}{-\frac{1}{2}} \ldots = \frac{(2\pi)^{\frac{1}{3}}}{2\sqrt{2}} B_{1\frac{1}{2}} \ldots$

multiplying the two results we have

$\frac{1}{2}\cdot\frac{1}{3} B_{-\frac{1}{2}} B_{\frac{1}{2}} = \frac{2\pi}{\ldots\sqrt{2}} \cdot \frac{1}{2\lfloor\frac{1}{3}} B_{\frac{1}{2}} B_{\frac{1}{2}}$ or $\frac{1}{3} = \frac{2\pi}{\ldots\lfloor\frac{1}{2}}$

$\therefore \lfloor\frac{1}{2}\lfloor\frac{1}{3} = \pi \ldots \lfloor\frac{1}{2} \ldots \sqrt{\pi}$

3. In a similar manner ... prove that $\lfloor n-1 \lfloor -n = \frac{\pi}{\ldots}$

4. Show that

$$\pi\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{6}} - \frac{1}{\sqrt{6}+\sqrt{8}} + \&c\right)$$

$$= \frac{1}{1\sqrt{1}} + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \frac{1}{7\sqrt{7}} + \&c$$

Sol. $L.H.S = \frac{\pi}{\sqrt{2}}\left\{(\sqrt{2}-\sqrt{1}) + (\sqrt{2}-\sqrt{4}) \ldots\right\}$

$= \pi\sqrt{2}(\sqrt{1}-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\&c)$ by VIII 5 ...

$= 2(2\sqrt{2}-1)\frac{B_{\frac{1}{2}}}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\frac{\pi}{\sqrt{2}}$ by IX 3 ...

$= \frac{8}{3}\frac{\pi}{\sqrt{2}}\left(1-\frac{1}{2\sqrt{2}}\right) \ldots \left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} \ldots \&c\right)$

$= \left(1-\frac{1}{2\sqrt{2}}\right)\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \&c\right) \ldots$

5. ~~$\pi\left(\frac{1}{\sqrt[3]{1}} - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} \ldots\right)$~~ $\lfloor\frac{1}{3}\lfloor\frac{1}{3}$

$= \left(\sqrt[3]{4\pi} + \sqrt[3]{2\pi}\right)\left(\frac{1}{\sqrt[3]{1}} - \frac{1}{\sqrt[3]{4}} + \ldots\right)$

6. Show that when x becomes infinite

i. $\sqrt{2(2x+1)} - \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots \frac{1}{\sqrt{x}}\right)$

$= (\sqrt{2}+1)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)$

ii. $\frac{2}{3}\sqrt{(x+\frac{1}{2})(x+\frac{1}{2})(x+\frac{3}{2})} - (\sqrt{1}+\sqrt{2}+\ldots+\sqrt{x})$

$= \frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \&c\right)$

iii. $\frac{2}{5}\sqrt{x(x+\frac{1}{2})(x+\frac{1}{2})(x+\frac{3}{2})(x+1)} + \frac{5}{768}(x+\frac{1}{2})$

$- (1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \ldots + x\sqrt{x})$

$= \frac{3}{16\pi^2}\left(\ldots + \frac{1}{\ldots\sqrt{2}} + \frac{1}{9\sqrt{3}} + \frac{1}{16\sqrt{4}} + \&c\right)$

$$\frac{\pi}{4}\sec\frac{\pi x}{2}\operatorname{Sech}\frac{\pi y}{2}$$

$$= \frac{1\operatorname{Sech}\frac{\pi x}{2y}}{1^2+y^2} - \frac{3\operatorname{Sech}\frac{3\pi x}{2y}}{3^2+y^2} + \frac{5\operatorname{Sech}\frac{5\pi x}{2y}}{5^2+y^2} - \&c$$

$$+ \frac{1\operatorname{Sech}\frac{\pi y}{2x}}{1^2-x^2} - \frac{3\operatorname{Sech}\frac{3\pi y}{2x}}{3^2-x^2} + \frac{5\operatorname{Sech}\frac{5\pi y}{2x}}{5^2-x^2} - \&c$$

$$\frac{\pi}{4}\sec\frac{\pi x}{2}\operatorname{Sech}\frac{\pi y}{2}\cdot\overline{\phi(xy)-\phi(-xy)}.$$

$$= \left\{\frac{\operatorname{Sech}\frac{\pi y}{2x}}{1^2-x^2}\cdot\overline{\phi(y)-\phi(-y)}\right.$$

$$= x\left\{\frac{\phi(y)-\phi(-y)}{1^2-x^2}\operatorname{Sech}\frac{\pi y}{2x} - \frac{\phi(3y)-\phi(-3y)}{3^2-x^2}\operatorname{Sech}\frac{3\pi y}{2x}\right.$$

$$\left. + \frac{\phi(5y)-\phi(-5y)}{5^2-x^2}\operatorname{Sech}\frac{5\pi y}{2x} - \&c\right\}$$

$$+ \frac{y}{i}\left\{\frac{\phi(xi)-\phi(-xi)}{1^2+y^2}\operatorname{Sech}\frac{\pi x}{2y} - \frac{\phi(3xi)-\phi(-3xi)}{3^2+y^2}\operatorname{Sech}\frac{3\pi x}{2y}\right.$$

$$\left. + \frac{\phi(5xi)-\phi(-5xi)}{5^2+y^2}\operatorname{Sech}\frac{5\pi x}{2y} - \&c\right\}$$

5. The following theorem is applied to [illegible] generally [illegible] will be found in [illegible]

If $x < 1$ the [illegible] of the general term [illegible] of the series [illegible]

$$x^n \phi(0) + \frac{n}{\lfloor 1} x^{n-1} \phi(1) + \frac{n(n-1)}{\lfloor 2} x^{n-2} \phi(2) + \&c = \phi(n)$$

Sol. The given series =

$$\phi(0) \left\{ x^n + \frac{n}{\lfloor 1} x^{n-1} + \frac{n(n-1)}{\lfloor 2} x^{n-2} + \&c \right\}$$

$$+ \phi'(0) \left\{ \frac{n}{\lfloor 1} x^{n-1} + [illegible] \frac{(n-1)}{\lfloor 2} x^{n-2} + \frac{3n(n-1)(n-2)}{\lfloor 3} x^{n-3} + \&c \right\}$$

$$+ \frac{1}{2} \phi''(0) \left\{ \frac{[illegible]}{\lfloor 1} [illegible] + \frac{[illegible](n-1)}{\lfloor 2} x^{n-2} + \frac{[illegible](n-1)(n-2)}{\lfloor 3} x^{n-3} + \&c \right\}$$

$+ \&c,$

$$= \phi(0)(1+x)^n + n\phi'(0)(1+x)^{n-1} + \frac{1}{2}\phi''(0)\left\{ n(n-1)(1+x)^{n-2} + n(1+x)^{n-1} \right\}$$

$$+ \&c = \phi(0) + \frac{n}{\lfloor 1}\phi'(0) + \frac{n^2}{\lfloor 2}\phi''(0) + \&c \quad \text{[illegible] } x = 0$$

$= \phi(n).$

Cor. When $x = 0$

$$\frac{\phi(1)}{x} - \frac{\phi(2)}{x^2} + \frac{\phi(3)}{x^3} - \frac{\phi(4)}{x^4} + \&c = \phi(0).$$

Sol. Write -1 for n in the above theorem.

Eg. Let $\phi(n) = \frac{1}{n} \sin \frac{\pi n}{2}$ then $\phi(0) = \frac{\pi}{2}$

when $x = 0$ $\quad \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \&c = \frac{\pi}{2}$

which is agreeable [illegible] $\tan^{-1} \infty = \frac{\pi}{2}$.

Ex. 1. If $x = 0$ Shew that

$$\frac{1}{\sqrt{x}} - \frac{1}{2} \frac{1}{x\sqrt{x}} + \frac{1.3.5}{2.4.6} \frac{1}{x^3\sqrt{x}} - \frac{1.3.5.7.[illegible]}{2.4.6.8.[illegible]} [illegible] = \sqrt{2}$$

2. If [illegible] then $\frac{1}{x} - \frac{\lfloor 1}{x^2} + \frac{\lfloor 2}{x^3} - \frac{\lfloor 3}{x^4} + \&c = \infty$. [illegible]

N.B. [illegible] $S = \frac{1}{x+1} - \frac{1^2}{x+2 - \frac{2^2}{x+3 - \frac{3^2}{x+4}}} = [illegible]$

$= 1 + [illegible] + \&c = \infty$

[illegible] $+ \frac{[illegible]}{\lfloor 1}(1+x) + \frac{[illegible]}{\lfloor 2}([illegible]) + \&c - [illegible]$

$= \infty$ [illegible] 0.

3. [illegible] $+ \frac{n}{\lfloor 1} x^{n-1} + n(n-1) x^{n-2} + $ [illegible]

$= \lfloor n$ [illegible]

$$\pi^2 x^2 \cot \pi x \coth \pi x$$
$$= 1 - 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 - x^4} + \frac{2\coth 2\pi}{2^4 - x^4} + \frac{3\coth 3\pi}{3^4 - x^4} + \&c \right\}.$$

$$\pi^2 xy \cot \pi x \coth \pi y$$
$$= 1 + 2\pi xy \left\{ \frac{\coth \frac{\pi x}{y}}{1^2 + y^2} + \frac{2\coth \frac{2\pi x}{y}}{2^2 + y^2} + \&c \right\}$$
$$- 2\pi xy \left\{ \frac{\coth \frac{\pi y}{x}}{1^2 - x^2} + \frac{2\coth \frac{2\pi y}{x}}{2^2 - x^2} + \&c \right\}$$

$$(\pi x)^2 \cdot \frac{\cosh \pi x\sqrt{2} + \cos \pi x\sqrt{2}}{\cosh \pi x\sqrt{2} - \cos \pi x\sqrt{2}}.$$
$$= 1 + 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 + x^4} + \frac{2\coth 2\pi}{2^4 + x^4} + \frac{3\coth 3\pi}{3^4 + x^4} + \&c \right\}$$

$$\frac{\pi}{8} \sec \frac{\pi x}{2} \operatorname{sech} \frac{\pi x}{2}$$
$$= \frac{1^3 \operatorname{sech} \frac{\pi}{2}}{1^4 - x^4} - \frac{3^3 \operatorname{sech} \frac{3\pi}{2}}{3^4 - x^4} + \frac{5^3 \operatorname{sech} \frac{5\pi}{2}}{5^4 - x^4} - \&c$$

$$\frac{\pi}{4} \cdot \frac{1}{\cosh \frac{\pi x}{\sqrt{2}} + \cos \frac{\pi x}{\sqrt{2}}}$$
$$= \frac{1^3 \operatorname{sech} \frac{\pi}{2}}{1^4 + x^4} - \frac{3^3 \operatorname{sech} \frac{3\pi}{2}}{3^4 + x^4} + \frac{5^3 \operatorname{sech} \frac{5\pi}{2}}{5^4 + x^4} - \&c$$

$$\int_0^\infty \frac{\cos nx}{4 + x^4}\, dx = \frac{\pi}{8} e^{-n} (\cos n + \sin n).$$

[illegible] $= 0$. Shew that [illegible]

$\frac{\ }{}\ [illegible] + \frac{[illegible]}{[illegible]}\ [illegible] = \frac{\pi}{2}$

Sol. write $\lfloor n-1$ [illegible] for $\phi(n)$ in the above [illegible]

then $\phi(0) = [illegible]$ [illegible] $= \frac{\pi}{2}$

N.B. Thus we are able to find exact values when [illegible] from the above theorem and even though the generating function [illegible] terms may be too difficult to find [illegible].

The generating function [illegible]

$= \frac{\pi}{2}\cos x + (c_0 + \log x)\sin x - [illegible]\left\{\frac{x}{[illegible]} - [illegible]\frac{x^3}{[illegible]} + [illegible]\right\}$

$= \frac{\pi}{2}$ when $x = 0$.

6. $(a+[illegible])^2 - (a+[illegible])^2 + (a+3[illegible])^2 - \&c = [illegible]\left\{\phi_2\left(\frac{[illegible]}{[illegible]}\right) - \phi_2\left(\frac{[illegible]}{[illegible]}\right)\right\}$

Ex. (a) If $x^2 + x = y$ and $x + \frac{1}{2} = a$ shew that

(1) $\phi_1(x) = \frac{y}{1}$; (2) $\phi_2(x) = [illegible]\frac{y}{3}$; (3) $\phi_3(x) = \frac{y^2}{[illegible]}$; (4) $\phi_4(x) = \frac{[illegible]}{5}y(y - [illegible])$

(5) $\phi_5(x) = \frac{y^2}{[illegible]}(y - [illegible])$; (6) $\phi_6(x) = \frac{[illegible]}{7}y(y^2 - [illegible]y + [illegible])$; (7) $\phi_7(x) = \frac{y^2}{[illegible]}(y^2 - [illegible]y + [illegible])$

(8) $\phi_8(x) = \frac{[illegible]}{9}y(y^3 - [illegible]y^2 + \frac{[illegible]}{[illegible]}y - \frac{[illegible]}{[illegible]})$; $\phi_9(x) = \frac{y^2}{[illegible]}(y^3 - [illegible]y^2 + [illegible]y - [illegible])$

(9) $\phi_9(x) = \frac{y^2}{10}(y-1)(y^2 - \frac{[illegible]}{2}y + \frac{[illegible]}{2})$; (11) $\phi_{10}(x) = \frac{[illegible]}{11}y(y-1)(y^2 - \frac{[illegible]}{[illegible]}y + [illegible])$

(12) $\phi_{11}(x) = \frac{y^2}{12}(y^4 - [illegible]y^3 + [illegible]y^2 - 10y + 5)$.

(b). 1. $y^2 = 4\phi_3(x) = 4\{\phi_1(x)\}^2$

2. $xy^2 = 5\phi_4(x) + \phi_2(x) = [illegible]\phi_1(x)\phi_2(x)$

3. $y^3 = 6\phi_5(x) + 2\phi_3(x) = 8\phi_1(x)\phi_3(x)$

3. $[illegible]y^3 = 7\phi_6(x) + 5\phi_4(x) = 12\phi_2(x)\phi_3(x)$

4. $[illegible] = [illegible]\phi_7(x) + 8\phi_5(x) = 16\phi_3(x)\phi_3(x)$

5. $[illegible] = [illegible] + [illegible]\phi_4(x) + \phi_2(x) = [illegible]\phi_2(x)\phi_3(x)$

6. $[illegible] = [illegible] + 20\phi_7(x) + 2\phi_5(x)$

7. $y^5 = [illegible] + 30\phi_8(x) + 7\phi_6(x)$

8. $xy^5 = [illegible] + [illegible] + 12\phi_7(x)$.

9. $y^6 = [illegible]$

7. [illegible] $\phi_2(x-1) + [illegible]\phi_2(x) = 0$.

$$\frac{\underline{\left|\dfrac{x+l+n-m-1}{2}\right.}\;\underline{\left|\dfrac{x+l-n-m-1}{2}\right.}\;\underline{\left|\dfrac{x-l+n+m-1}{2}\right.}\;\underline{\left|\dfrac{x-l-n+m-1}{2}\right.}}{\underline{\left|\dfrac{x-l+n-m-1}{2}\right.}\;\underline{\left|\dfrac{x-l-n-m-1}{2}\right.}\;\underline{\left|\dfrac{x+l+n+m-1}{2}\right.}\;\underline{\left|\dfrac{x+l-n+m-1}{2}\right.}} = P$$

then $$\frac{1-P}{1+P} = \frac{2lmx}{x^2+l^2+m^2-n^2-1} + \frac{4(x^2-1)(l^2-1)(m^2-1)}{3(x^2+l^2+m^2-n^2-5)} + \&c$$

The expansion only is true.

$$\frac{1}{x^2} - \frac{2\cos\theta}{1^2-x^2} - \frac{2\cos 2\theta}{2^2-x^2} - \frac{2\cos 3\theta}{3^2-x^2} - \&c$$

$$= \frac{\pi}{x}\left\{\cot \pi x \cos \theta x + \sin \theta x\right\}$$

$$\frac{\sin\theta}{1^2-x^2} - \frac{\sin 3\theta}{3^2-x^2} + \frac{\sin 5\theta}{5^2-x^2} - \&c = \frac{\pi}{4x}\sec\frac{\pi x}{2}\sin\theta x$$

Sol. Let $\psi(x) = \phi_n(x-1) - (-1)^n \phi_n(-x)$ then we see that $\psi(x+1) - \psi(x) = 0$ [illegible] if n is a positive integer $\psi(x) = 0$ [illegible] that $\phi_n(x-1) = \phi_n(-x)$.

Ex. 1. [illegible] is divisible by $x^2(x+1)^2$ [illegible] $x(x+\frac{1}{2})(x+1)$

2. Show that $\phi_n(x)$ is divisible by [illegible] according as n is odd or even and point out the excep[tion] [illegible] in both cases n being a positive integer.

8. $\phi_n(x) = -B_n x \cos\frac{\pi n}{2} - \frac{n}{\lfloor 2} B_{n-1} x^2 \sin\frac{\pi n}{2} + \frac{n(n-1)}{\lfloor 3} B_{n-2} x^3 \cos\frac{\pi n}{2}$

$+ \frac{n(n-1)(n-2)}{\lfloor 4} B_{n-3} x^4 \sin\frac{\pi n}{2} - \&c$

$= -nxS_{1-n} - \frac{n(n-1)}{\lfloor 2} x^2 S_{2-n} + \frac{n(n-1)(n-2)}{\lfloor 3} x^3 S_{3-n} - \&c.$

Sol. Apply VIII 3.

9. $\phi_n(x) = 1^n - (1+x)^n + 2^n - (2+x)^n + 3^n - (3+x)^n + \&c.$

10. $\phi_n(x) - n^n \left\{ \phi_n\left(\frac{x}{n}\right) + \phi_n\left(\frac{x-1}{n}\right) + \phi_n\left(\frac{x-2}{n}\right) + \dots + \phi_n\left(\frac{x-n+1}{n}\right) \right\}$

$= (n^{n+1} - 1) \frac{B_{n+1}}{n+1} \sin\frac{\pi n}{2} = (1 - n^{n+1}) S_{-n}$

Sol. Apply VIII 4.

Cor. $\phi_n(-\frac{1}{n}) + \phi_n(-\frac{2}{n}) + \phi_n(-\frac{3}{n}) + \dots + \phi_n(-\frac{n-1}{n}) = (n - n^{-n}) S_{-n}$

$= (n^{-n} - n) \frac{B_{n+1}}{n+1} \sin\frac{\pi n}{2}$.

11. If n is a negative integer, then

$\phi_n(x-1) + (-1)^n \phi_n(-x) = \{1 + (-1)^n\} S_n + \frac{(-1)^n \, d^{-(n+1)x} \pi \cot \pi x}{\lfloor -n-1}$

Sol. From X we have $\phi_{-1}(x-1) - \phi_{-1}(-x) = -\pi \cot \pi x$

Differentiate both sides n times.

N.B. The following method is very useful in finding the [illegible] of $\pi \cot \pi x$. Let $\cot \pi x = y$. The [illegible] of the coeff. of π^n are the same as those in [illegible] of [illegible]

Each [illegible] divisible by y^2+1 so that the table [illegible] easily formed.

$$\left\{ \left\lfloor \frac{\alpha+\beta+\gamma+\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta+\gamma-\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta-\gamma+\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta+\gamma+\delta+\epsilon-1}{2} \right. \times \right.$$

$$\left. \left\lfloor \frac{\alpha+\beta-\gamma-\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta+\gamma-\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta-\gamma+\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta-\gamma-\delta+\epsilon-1}{2} \right. \right\} = Q$$

$$\left\{ \left\lfloor \frac{\alpha+\beta+\gamma+\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta+\gamma-\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta-\gamma-\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta-\gamma+\delta+\epsilon-1}{2} \right. \times \right.$$

$$\left. \left\lfloor \frac{\alpha-\beta+\gamma+\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta+\gamma-\delta+\epsilon-1}{2} \right. \left\lfloor \frac{\alpha+\beta-\gamma+\delta-\epsilon-1}{2} \right. \left\lfloor \frac{\alpha-\beta-\gamma-\delta-\epsilon-1}{2} \right. \right\} = P$$

$$\neq \frac{8\alpha\beta\gamma\delta\epsilon}{\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1) - (\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-1)^2 - 2^2\right\}} +$$

$$\frac{64(\alpha^2-1)(\beta^2-1)(\gamma^2-1)(\delta^2-1)(\epsilon^2-1)}{3\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1) - (\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-5)^2 - 6^2\right\}} + \&c$$

$= \frac{P-Q}{P+Q}$. if any one of $\alpha, \beta, \gamma, \delta, \epsilon$ be an integer

πy Write under each [illegible] term the quotient obtained by
$\pi^2(y^2+1)$ dividing [illegible] of the product of the coeff.t
$\pi^3(y^3+y)$ [illegible] of that term and of the
$\pi^4(y^4+\frac{2}{3}y^2+\frac{1}{3})$ [illegible] number of [illegible] by the coeff.t of π.
[illegible] term [illegible]

$\pi^5(y^5+\frac{5}{3}y^3+\frac{2}{3}y)$

$\pi^6(y^6+2y^4+\frac{17}{15}y^2+\frac{2}{15})$

$\pi^7(y^7+\frac{7}{3}y^5+\frac{7}{5}y^3+\frac{17}{45}y)$

$\pi^8(y^8+\frac{8}{3}y^6+\frac{[illegible]}{[illegible]}y^4+\frac{248}{315}y^2+\frac{17}{315})$

$\pi^9(y^9+3y^7+\frac{16}{5}y^5+\frac{[illegible]}{63}y^3+\frac{62}{315}y)$

$\pi^{10}(y^{10}+\frac{10}{3}y^8+\frac{37}{9}y^6+\frac{424}{189}y^4+\frac{1382}{2835}y^2+\frac{62}{2835})$

Ex. (a) For all values of n shew that

1. $\phi_n(x) - 2^n\left\{\phi_n\left(\frac{x}{2}\right)+\phi_n\left(\frac{x-1}{2}\right)\right\} = (1-2^{n+1})S_{-n}$
2. $\phi_n(-\frac{1}{2}) = (2-\frac{1}{2^n})S_{-n}$
3. $\phi_n(-\frac{1}{3}) + \phi_n(-\frac{2}{3}) = (3-\frac{1}{3^n})S_{-n}$
4. $\phi_n(-\frac{1}{4}) + \phi_n(-\frac{3}{4}) = (4+\frac{1}{2^n}-\frac{1}{4^n})S_{-n}$
5. $\phi_n(-\frac{1}{6}) + \phi_n(-\frac{5}{6}) = (1+\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{6^n})S_{-n}$

(b) If n is a positive odd integer shew that

1. $\phi_n(-\frac{1}{3}) = (2-\frac{1}{3^n})\frac{S_{-n}}{2}$
2. $\phi_n(-\frac{1}{4}) = (1+\frac{1}{2^{n+1}}-\frac{1}{2^{2n+1}})S_{-n}$
3. $\phi_n(-\frac{1}{6}) = (1+\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{6^n})\frac{S_{-n}}{2}$
4. $\phi_n(-\frac{1}{5}) + \phi_n(-\frac{2}{5}) = (5-\frac{1}{5^n})\frac{S_{-n}}{2}$
5. $\phi_n(-\frac{1}{8}) + \phi_n(-\frac{3}{8}) = (2+\frac{1}{2^{2n+1}}-\frac{1}{2^{3n+1}})S_{-n}$
6. $\phi_n(-\frac{1}{10}) + \phi_n(-\frac{3}{10}) = (5+\frac{1}{5^n}-\frac{1}{10^n})\frac{S_{-n}}{2}$
7. $\phi_n(-\frac{1}{12}) + \phi_n(-\frac{5}{12}) = (6+\frac{1}{6^n}-\frac{1}{12^n})\frac{S_{-n}}{2}$

(c) Show that [illegible] are even for positive integral values [illegible]
and hence [illegible] from 18 ||.

$$\frac{\phi(\log 1)}{1} + \frac{\phi(\log 2)}{2} + \frac{\phi(\log 3)}{3} + \cdots$$

$$= \int_0^\infty \phi(x)\,dx + c_0\phi(0) + \frac{c_1}{\underline{1}}\phi'(0) + \cdots$$

$$2x + \frac{2^x - 2^{-x}}{1\cdot 2} + \frac{3^x - 3^{-x}}{2\cdot 3} + \frac{4^x - 4^{-x}}{3\cdot 4} + \&c$$

$$= \frac{1}{x} - \pi\cot\pi x + \&c$$

$$\frac{1}{2(2^x-1)} + \frac{1}{3(3^x-1)} + \frac{1}{4(4^x-1)} + \cdots$$

$$= \frac{.7946786 - \log_e x}{x} + .2113922$$

$$- .0060680\,x - .0000028\,x^3 + \&c$$

$$\frac{1}{2\log_e 2} + \frac{1}{3\log_e 3} + \frac{1}{4\log_e 4} + \cdots + \frac{1}{x\log_e x}$$

$$= .7946786 + \log_e\log_e(x+\tfrac{1}{2}) \text{ nearly.}$$

$$\frac{1}{2^{x+1}\log_e 2} + \frac{1}{3^{x+1}\log_e 3} + \frac{1}{4^{x+1}\log_e 4} + \cdots$$

$$= ~~.7946786~~ - \log_e x + .4227843~~351~~\,x$$

$$- .0364079~~2274~~\,x^2 + .001617\,x^3$$

$$+ .000085\,x^4 - .00002\,x^5 - \&c$$

$$+ .2174630$$

(d). 1. $\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+$ &c $=\frac{7}{8}S_3$.

2. $\frac{1}{1^3}+\frac{1}{4^3}+\frac{1}{7^3}+\frac{1}{10^3}+$ &c $=\frac{2\pi^3}{81\sqrt{3}}+\frac{13}{27}S_3$.

3. $\frac{1}{1^3}+\frac{1}{5^3}+\frac{1}{9^3}+\frac{1}{13^3}+$ &c $=\frac{\pi^3}{64}+\frac{7}{16}S_3$

4. $\frac{1}{1^3}+\frac{1}{7^3}+\frac{1}{13^3}+\frac{1}{19^3}+$ &c $=\frac{\pi^3}{36\sqrt{3}}+\frac{91}{216}S_3$.

12. $2^n\left\{\phi_n(-\frac{1}{6})-\phi_n(-\frac{5}{6})\right\}=(2^n+1)\left\{\phi_n(-\frac{1}{3})-\phi_n(-\frac{2}{3})\right\}$.

Sol. $\phi_n(-\frac{1}{3})-2^n\left\{\phi_n(-\frac{1}{6})+\phi_n(-\frac{2}{3})\right\}=(2^{n+1}-1)S_{-n}$
& $\phi_n(-\frac{2}{3})-2^n\left\{\phi_n(-\frac{1}{3})+\phi_n(-\frac{5}{6})\right\}=(2^{n+1}-1)S_{-n}$ } by VIII 10.

$\therefore 2^n\left\{\phi_n(-\frac{1}{6})-\phi_n(-\frac{5}{6})\right\}=(2^n+1)\left\{\phi_n(-\frac{1}{3})-\phi_n(-\frac{2}{3})\right\}$.

13. If C_n be the constant of $\frac{(\log 1)^n}{1}+\frac{(\log 2)^n}{2}+\frac{(\log 3)^n}{3}+$ &c then

$S_{n+1}=\frac{1}{n}+C_0-\frac{n}{1!}C_1+\frac{n^2}{2!}C_2-\frac{n^3}{3!}C_3+$ &c.

Sol. It has been proved in VIII 6 ex 1. that $S_{n+1}-\frac{1}{n}$ is finite when $n=0$. The remaining part is obtained by applying VIII 6 Case II.

The above result may be written as follows

$\frac{1}{1^{n+1}}+\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}+\frac{1}{4^{n+1}}+$ &c

$=\frac{1}{n}+.5772156649+.0728158455\,n$

$-(.00485\,n^2+.000342\,n^3)+\frac{n^4}{[illegible]}\theta$.

where θ may be taken as equivalent to $1+\frac{2n}{50}$.

N.B. The above result is true for all values of n.

Ex. 1. $\frac{1}{1\sqrt[10]{1}}+\frac{1}{2\sqrt[10]{2}}+\frac{1}{3\sqrt[10]{3}}+$ &c $=10.5844843$

2. $\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+$ &c $=2.612375$.

3. $\frac{1}{1^2\sqrt{1}}+\frac{1}{2^2\sqrt{2}}+\frac{1}{3^2\sqrt{3}}+$ &c $=1.341490$.

4. $B_{\frac{1}{2}}=-.4409932$ 5. $B_{\frac{1}{4}}=-1.032627$

6. $B_{\frac{1}{3}}=-.9420765$ & $B_{-\frac{1}{2}}=-1.3841347$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^{n-h}dx$$

$$= 1+\left(1-\frac{h}{n}\right)+\left(1-\frac{h}{n}\right)\left(1-\frac{h+1}{n}\right)+\left(1-\frac{h}{n}\right)\left(1-\frac{h+1}{n}\right)\left(1-\frac{h+2}{n}\right) + \&c$$

$$= \frac{e^n \lfloor n-h}{2n^{n-h}} + A_0 - \frac{A_1}{n} + \frac{A_2}{n^2} - \&c$$

$$A_0 = \frac{2}{3} - h. \quad A_1 = \frac{4}{135} - \frac{h^2(1-h)}{3}$$

$$A_2 = \frac{8}{2835} + \frac{2h(1-h)}{135} - \frac{h(1-h^2)(2-3h^2)}{45}. \quad \&c$$

$$1 + \frac{\phi(h.\overline{\alpha+\delta})}{\phi(h.\overline{\beta+\gamma})} + \frac{\phi(h.\overline{\alpha+\delta})\,\phi(h.\overline{\alpha+2\delta})}{\phi(h.\overline{\beta+\gamma})\,\phi(h.\overline{\beta+2\gamma})} + \&c.$$

$$= \sqrt{\frac{\pi\,\phi(0)}{2(\gamma-\delta)h\,\phi'(0)}} + \frac{1}{3}\cdot\frac{\gamma+\delta}{\gamma-\delta}\left\{1-\frac{\phi(0)}{\phi'(0)}\cdot\frac{\phi''(0)}{\phi'(0)}\right\} + \frac{\alpha-\beta}{\gamma-\delta}.$$

if h is very small.

$$1 + \frac{x^2}{2}\cdot\frac{1}{2n+1} + \frac{x^4}{2.4}\cdot\frac{1}{(2n+1)(2n+3)} + \&c$$

$$= \frac{2^{n-1}\lfloor n-\frac{1}{2}}{x^n\sqrt{\pi}}\left[e^x\left\{1-\frac{n(n-1)}{2}\cdot\frac{1}{x}+\frac{(n+1)n(n-1)(n-2)}{2.4}\frac{1}{x^2} - \&c\right\} + \theta e^{-x}\cos\pi n\right]$$

where $\theta = e^{\frac{n(n-1)}{2x+1}}$ *nearly.*

7. $B_{-\frac{1}{2}} \frac{\ }{\ }$ -1.847228

8. Show that $S_{1+n} + S_{1-n} = \dfrac{2S_0}{1 + \cdot 00839n^2 + \cdot 0001n^4 + \&c}$

14. $\dfrac{\phi_n(x-1) - \phi_n(-x)}{4\underline{|n}} = -\cos\dfrac{\pi n}{2}\left\{\dfrac{\sin 2\pi x}{(2\pi)^{n+1}} + \dfrac{\sin 4\pi x}{(4\pi)^{n+1}} + \&c\right\}$

Sol. $\phi_n(x-1) - \phi_n(-x)$
$= (1-x)^n - x^n + (2-x)^n - (1+x)^n + (3-x)^n - (2+x)^n + \&c$ and
Then arrange the terms in ascending powers of x and
Substitute $\dfrac{B_n}{n}\cos\dfrac{\pi n}{2}$ for S_{1-n}. Similarly

15. $\dfrac{\phi_n(x-1) + \phi_n(-x) - 2S_{-n}}{4\underline{|n}} = \sin\dfrac{\pi n}{2}\left\{\dfrac{\cos 2\pi x}{(2\pi)^{n+1}} + \dfrac{\cos 4\pi x}{(4\pi)^{n+1}} + \&c\right\}$

N.B. The above two theorems are true for all values of x when n is an integer but when n is fractional they are true only when x lies between 0 and 1.

16. $\dfrac{(2\pi q)^n}{4\underline{|n-1}}\left\{\phi_{n-1}\left(\dfrac{p}{q}-1\right) - \phi_{n-1}\left(\dfrac{p}{q}\right)\right\}$ when $\dfrac{p}{q}$ lies between 0 & 1

$= -\sin\dfrac{\pi n}{2}\left[\left\{S_n - \phi_{-n}\left(\dfrac{1}{q}-1\right)\right\}\sin\dfrac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\dfrac{2}{q}-1\right)\right\}\sin\dfrac{4\pi p}{q}\right.$
$\left. + \left\{S_n - \phi_{-n}\left(\dfrac{3}{q}-1\right)\right\}\sin\dfrac{6\pi p}{q} + \dots + \left\{S_n - \phi_{-n}\left(\dfrac{q-1}{q}-1\right)\right\}\sin\dfrac{(2q-2)\pi p}{q}\right]$

17. $\dfrac{(2\pi q)^n}{4\underline{|n-1}}\left\{\phi_{n-1}\left(\dfrac{p}{q}-1\right) + \phi_{n-1}\left(-\dfrac{p}{q}\right) - 2S_{1-n}\left(1-\dfrac{1}{q^n}\right)\right\}$ for the same limits

$= -\cos\dfrac{\pi n}{2}\left[\left\{S_n - \phi_{-n}\left(\dfrac{1}{q}-1\right)\right\}\cos\dfrac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\dfrac{2}{q}-1\right)\right\}\cos\dfrac{4\pi p}{q}\right.$
$\left. + \left\{S_n - \phi_{-n}\left(\dfrac{3}{q}-1\right)\right\}\cos\dfrac{6\pi p}{q} + \dots + \left\{S_n - \phi_{-n}\left(\dfrac{q-1}{q}-1\right)\right\}\cos\dfrac{(2q-2)\pi p}{q}\right]$

In the above two theorems p & q are integers.
In Ex. 16 & 17 the same theorems 14 & 15 are written in another form.

Ex. 1. $\phi_n\left(-\dfrac{1}{4}\right) - \phi_n\left(-\dfrac{3}{4}\right) = 2\cdot\dfrac{E_{n+1}}{4^{n+1}}\cos\dfrac{\pi n}{2}$

Sol. Use [illegible] for x in IX 14.

$$1 + \frac{(ex)}{1} + \frac{(ex)^2}{2^2} + \frac{(ex)^3}{3^3} + \&c$$

$$= \sqrt{2\pi x}\, e^{x - \frac{1}{24x} - \frac{1}{48x^2} - \left(\frac{1}{36} + \frac{1}{5760}\right)\frac{1}{x^3}}$$

$$\int_0^\infty \frac{x^{n-1}\,dx}{1 + \left(\frac{x}{1}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{3}\right)^3 + \left(\frac{x}{4}\right)^4 + \&c}$$

$$= n^n \left\{ \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \frac{3}{8n^4} + \&c \right\}$$

2. $1^n - 3^n + 5^n - 7^n + \&c = \frac{1}{2} E_{n+1} \cos\frac{\pi n}{2}$.

3. $E_{1-n} \cos\frac{\pi n}{2} = \left(\frac{\pi}{2}\right)^n \frac{E_n}{\lfloor n}$

4. $\pi\left\{\frac{1}{2} - \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} - \frac{1}{\sqrt{5}+\sqrt{7}} + \&c\right\}$

$= \frac{1}{1\sqrt{1}} - \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} - \frac{1}{7\sqrt{7}} + \&c.$

5. $\frac{\pi}{2}\left\{\frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{\sqrt{1}+\sqrt{3}} - \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{4}+\sqrt{6}} - \&c\right\}$

$= \frac{1}{1\sqrt{1}} + \frac{1}{5\sqrt{5}} + \frac{1}{9\sqrt{9}} + \frac{1}{13\sqrt{13}} + \&c$

6. If $\frac{p}{q}$ lies between 0 & 1, p being any integer & q an odd integer, shew that

$\frac{(2\pi q)^n}{2\lfloor n-1}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right) - \phi_{n-1}\left(-\frac{p}{q}\right)\right\}$

$= \sin\frac{\pi n}{2}\Big[\left\{\phi_{-n}\left(\frac{p}{q}-1\right) - \phi_{-n}\left(-\frac{p}{q}\right)\right\}\sin\frac{2\pi p}{q}$

$+ \left\{\phi_{-n}\left(\frac{2p}{q}-1\right) - \phi_{-n}\left(-\frac{2p}{q}\right)\right\}\sin\frac{4\pi p}{q} + \&c \text{ to } \frac{q-1}{2} \text{ terms}\Big]$

7. $\frac{(2\pi q)^n}{2\lfloor n-1}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right) + \phi_{n-1}\left(-\frac{p}{q}\right) - 2S_{1-n}\left(1-\frac{1}{q^n}\right)\right\}$

$= \cos\frac{\pi n}{2}\Big[\left\{\phi_{-n}\left(\frac{p}{q}-1\right) - \phi_{-n}\left(-\frac{p}{q}\right)\right\}\cos\frac{2\pi p}{q} + \left\{\phi_{-n}\left(\frac{2p}{q}-1\right) - \phi_{-n}\left(-\frac{2p}{q}\right)\right\}\cos\frac{4\pi p}{q}$

$+ \&c \text{ to } \frac{q-1}{2} \text{ terms}\Big]$

Hence shew that

$\frac{(6\pi)^n}{2\lfloor n-1\sqrt{3}}\left\{\phi_{n-1}\left(-\frac{2}{3}\right) - \phi_{n-1}\left(-\frac{1}{3}\right)\right\} = \left\{\phi_{-n}\left(-\frac{2}{3}\right) - \phi_{-n}\left(-\frac{1}{3}\right)\right\}\sin\frac{\pi n}{2}$

8. $\frac{2^{n-1}}{\lfloor n}\phi_n(-x) = \frac{\sin \pi x}{\pi^{n+1}}\cos\left(\pi x + \frac{\pi n}{2}\right) + \frac{\sin 2\pi x}{(2\pi)^{n+1}}\cos\left(2\pi x + \frac{\pi n}{2}\right) + \&c$

Combine the results of IX 11 & 15.

9. If $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{1-x}} + \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{2-x}} + \frac{1}{\sqrt{2+x}} - \&c$

$= 2\left(\frac{\sin 2\pi x}{\sqrt{1}} + \frac{\sin 4\pi x}{\sqrt{2}} + \frac{\sin 6\pi x}{\sqrt{3}} + \&c\right)$

$$(p-q-1)\int_0^\infty \frac{\left(1+\frac{x}{n}\right)^q}{\left(1+\frac{x}{m}\right)^p}\,dx$$

$$= \frac{1}{2}\cdot\frac{m^p}{n^q}\cdot\frac{\lfloor q}{\lfloor p-1}\cdot\frac{\lfloor p-q-1}{(m-n)^{p-q-1}}$$

$$+ \frac{2}{3}(m+n) + mq - np$$

CHAPTER X

1. $\frac{B_r}{r}\cos\frac{\pi r}{2} + \frac{1}{r}$ when r is 0 is a finite quantity which is invariably denoted by the symbol C_0 the value of which can be found from IX 2. It is the constant of S_1 & its value is $\cdot 5772156649015328606\ldots$ & $e^{-C_0} = \cdot 56145,948356$

Sol. Since L.H.S in IX 1 is finite when $r = -1$

$\frac{B_r}{r}\cos\frac{\pi r}{2} + \frac{x^r}{r}$ is finite when $r = 0$ i.e.

$\frac{B_r}{r}\cos\frac{\pi r}{2} + \frac{1}{r} + \frac{x^r - 1}{r}$ is finite when $r = 0$

But $\frac{x^r - 1}{r} = \log_e x$ a finite quantity when $r = 0$

$\therefore \frac{B_r}{r}\cos\frac{\pi r}{2} + \frac{1}{r}$ is finite when $r = 0$.

2. $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{x} = \Sigma\frac{1}{x} = \phi(x)$.

$\Sigma\frac{1}{x} = C_0 + \log_e x + \frac{1}{2x} - \frac{B_2}{2x^2} + \frac{B_4}{4x^4} - \frac{B_6}{6x^6} + \frac{B_8}{8x^8} - \&c$

3. $\Sigma\frac{1}{x} = 1 - \frac{1}{x+1} + \frac{1}{2} - \frac{1}{x+2} + \frac{1}{3} - \frac{1}{x+3} + \&c$

$= \frac{x}{1(1+x)} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \&c$

4. $\Sigma\frac{1}{x} = xS_2 - x^2S_3 + x^3S_4 - x^4S_5 + x^5S_6 - \&c$

5. $\Sigma\frac{1}{x-1} - \Sigma\frac{1}{-x} = -\pi\cot\pi x$.

Sol. write πx for x in . Then we have

$\pi\cot\pi x = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \&c$

$= \left\{1 - \frac{1}{1-x} + \frac{1}{2} - \frac{1}{2-x} + \frac{1}{3} - \frac{1}{3-x} + \&c\right\}$

$- \left\{1 - \frac{1}{x} + \frac{1}{2} - \frac{1}{1+x} + \frac{1}{3} - \frac{1}{2+x} + \&c\right\}$

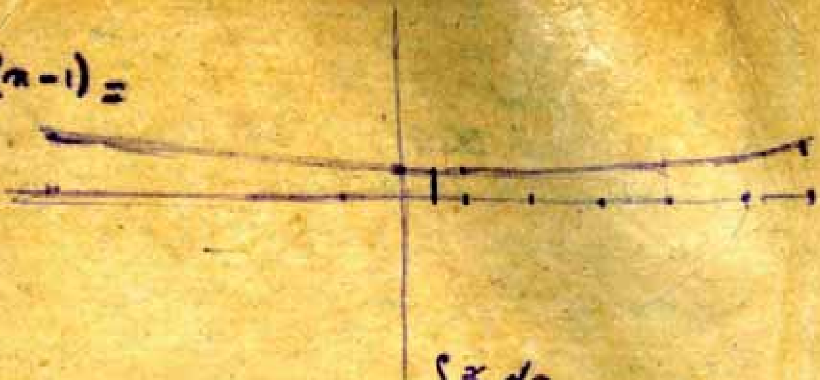
$= \Sigma\frac{1}{-x} - \Sigma\frac{1}{x-1}$ by X 3.

6. $n\,\Sigma\frac{1}{x} - \left\{\Sigma\frac{1}{x/n} + \Sigma\frac{1}{x - \frac{1}{n}} + \Sigma\frac{1}{x - \frac{2}{n}} + \ldots + \Sigma\frac{1}{x - \frac{n-1}{n}}\right\}$

$= n\log_e n$.

Ex 1. $\Sigma\frac{1}{x-\frac{1}{2}} = C_0 + \log_e x + \frac{2-1}{2}\frac{B_2}{2x^2} - \frac{2^3-1}{2^3}\frac{B_4}{4x^4} + \frac{2^5-1}{2^5}\frac{B_6}{6x^6} - \&c$

$$\frac{\lfloor x}{\pi^{\frac{x}{2}}} S_x(x-1) =$$



The maximum value of $\frac{a^x}{\lfloor x} = \frac{e^{\int \frac{x}{a} da}}{\sqrt{2\pi}}$

$$= \frac{a^{a-\frac{1}{2}}}{\lfloor a-\frac{1}{2}} e^{\frac{1}{32a(36a^2+10\cdot 1)}}$$

2. $x\frac{1}{\frac{1}{n}} + x^2\frac{1}{\frac{-2}{n}} + x^3\frac{1}{\frac{-3}{n}} + \cdots = \frac{1}{\frac{-n+1}{n}} = -n\log_e n$

3. (a) $\phi(-\frac{1}{2}) = -2\log_e 2$, (b) $\phi(-\frac{2}{3}) = -\frac{3}{2}\log_e 3 - \frac{\pi}{2\sqrt{3}}$

(c) $\phi(\frac{3}{4}) = -\frac{\pi}{2} - 3\log_e 2$ (d) $\phi(-\frac{5}{6}) = -\frac{\pi}{2}\sqrt{3} - 2\log_e 2 - \frac{3}{2}\log_e 3$

(e) $3\phi(-\frac{1}{3}) - 2\phi(-\frac{1}{2}) = \pi$

4. $\phi(\frac{1}{2n}) + \phi(-\frac{3}{2n}) + \cdots \cdots + \phi(-\frac{2n-1}{2n}) = -n\log_e 4n$.

7. $\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots + \frac{1}{a+xb} = \frac{1}{b}\{\phi(\frac{a}{b}+x) - \phi(\frac{a}{b})\}$

8. $\frac{1}{a+b} - \frac{1}{a+2b} + \frac{1}{a+3b} - \&c = \frac{1}{2b}\{\phi(\frac{a}{2b}) - \phi(\frac{a-b}{2b})\}$

9. $\phi(\frac{1}{2x}) = \phi(\frac{1}{x}) - \log_e 2 + x\int_0^1 \frac{n^x}{1+n^x}\,dn$

10. $\phi(-\frac{1}{2}) = -2\int_0^1 \frac{(1-n)^2}{n(n^2-1)}\,dn$.

11. $\phi(\frac{1}{x}-1) + \phi(-\frac{1}{x}) = -x\{1 + \frac{2}{x^2-x} + \frac{2}{(2x)^2-2x} + \&c\}$

12. $\frac{2}{x^2-x} + \frac{2}{(2x)^2-2x} + \frac{2}{(3x)^2-3x} + \&c = \int_0^1 \frac{n^{x-2}(1-n)^2}{1-n^x}\,dn$

13. $1 + \frac{2}{(2x)^3-2x} + \frac{2}{(4x)^3-4x} + \frac{2}{(6x)^3-6x} + \&c$

$= \frac{1}{x}\{1 + \frac{2}{(2x)^2-x} + \frac{2}{(2x)^3-2x} + \&c\} + \frac{\log_e 2}{x} \ldots$

$+$ log mod part of $(1 + \frac{1}{1+x} + \frac{1}{1+2x} + \frac{1}{1+3x} + \&c)$

14. (a) $\frac{x}{1} - \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} - \&c = \int_0^x \frac{1}{1+x^n}\,dx$

(b) $\frac{x}{1} + \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} + \&c = \int_0^x \frac{1}{1-x^n}\,dx$

(c) If n is odd $\int_0^x \frac{1}{1-x^n}\,dx = \int_0^{-x} \frac{1}{1+(-x)^n}\,dx$

(d) If n is even $\int_0^x \frac{1}{1-x^n}\,dx = \frac{1}{2}\int_0^x \frac{1}{1+x^{\frac{n}{2}}}\,dx + \frac{1}{2}\int_0^x \frac{1}{1-x^{\frac{n}{2}}}\,dx$

(e) If $l < n+1$

1. If n is even $\int \frac{x^{l-1}}{x^n-1}\,dx = \frac{1}{n}\log_e(x-1) + \frac{(-1)^l}{n}\log_e(x+1)$

$+ \frac{1}{n}\sum \cos\frac{lk\pi}{n}\log_e(x^2 - 2x\cos\frac{k\pi}{n} + 1)$

$- \frac{2}{n}\sum \sin\frac{lk\pi}{n}\tan^{-1}\frac{x - \cos\frac{k\pi}{n}}{\sin\frac{k\pi}{n}}$

$$\int_0^\infty \frac{x^{2m}}{(1+x^2)^{n+1}} \cos px\, dx = \frac{\pi}{2}(-1)^m \frac{e^{-p}}{2^n \lfloor n}$$

$$\times \left\{ p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \&c \right\}$$

$$A_r = \frac{\lfloor n+r}{\lfloor n-r} \cdot \frac{1}{2^r \lfloor r} \left\{ 1 - \frac{4}{\lfloor 1} \cdot \frac{rmn}{(n+r)(n+r-1)} \right.$$

$$\left. + \frac{4^2}{\lfloor 2} \cdot \frac{r(r-1)\, m(m-1)\, n(n-1)}{(n+r)(n+r-1)(n+r-2)(n+r-3)} - \&c \right\}$$

$$\cot\theta + 4\left(\frac{\sin 2\theta}{e^{2y}-1} + \frac{\sin 4\theta}{e^{4y}-1} + \frac{\sin 6\theta}{e^{6y}-1} + \&c \right)$$

~~$= 2\cot\phi\sqrt{1-x\sin^2\phi}$~~ $+ z\int_0^\phi \sqrt{1-x\sin^2\phi}\, d\phi$

~~$- \frac{2\theta z^2}{\pi}\int_0^{\frac{\pi}{2}}$~~

$$= z\left\{ \cot\phi\sqrt{1-x\sin^2\phi} + \int_0^\phi \sqrt{1-x\sin^2\phi}\, d\phi \right.$$

$$\left. - \frac{2\theta}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-x\sin^2\phi}\, d\phi \right\}$$

$$4\left(\frac{\sin 2\theta}{e^{y}-e^{-y}} + \frac{\sin 4\theta}{e^{2y}-e^{-2y}} + \&c \right)$$

$$= z\left\{ \int_0^\phi \sqrt{1-x\sin^2\phi}\, d\phi - \frac{2\theta}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1-x\sin^2\phi}\, d\phi \right\}$$

ii. If n be odd $\int \frac{x^{l-1}}{x^n-1}dx = \frac{1}{n}\log_e(x-1) + \frac{1}{n}\sum \cos\frac{rl\pi}{n}\log_e(x^2-2x\cos\frac{r\pi}{n}+1)$

$- \frac{2}{n}\sum \sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{r\pi}{n}}$ $r = 2, 4, 6, \ldots (n-1)$

iii. If n be even $\int \frac{x^{l-1}}{x^n+1}dx = -\frac{1}{n}\sum \cos\frac{rl\pi}{n}\log_e(x^2-2x\cos\frac{r\pi}{n}+1)$

$+ \frac{2}{n}\sin\frac{\pi rl}{n}\tan^{-1}\frac{x-\cos\frac{\pi r}{n}}{\sin\frac{\pi r}{n}}$ $r = 1, 3, 5, \ldots (n-1)$

iv. If n be odd $\int \frac{x^{l-1}}{x^n+1}dx = \frac{(-1)^{l-1}}{n}\log_e(x+1)$

$- \frac{1}{n}\sum \cos\frac{rl\pi}{n}\log_e(x^2-2x\cos\frac{r\pi}{n}+1)$

$+ \frac{2}{n}\sum \sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{r\pi}{n}}$ $r = 1, 3, 5, \ldots (n-2)$

v. $\int_a^b f(x)dx = h\left\{\frac{1}{2}f(b) + \frac{1}{2}f(a) + f(a+h) + \ldots + f(b-h)\right\}$

$- \frac{h^2}{\underline{2}}B_1\left\{f'(b) - f'(a)\right\} + \frac{h^4}{\underline{4}}B_2\left\{f'''(b) - f'''(a)\right\} - \&c$

Ex 1. $\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \&c = \log_e(1+x)$

2. $\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \&c = \tan^{-1}x$

3. $\frac{x}{1} - \frac{x^4}{4} + \frac{x^7}{7} - \&c = \frac{1}{6}\log_e\frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2-x}$

4. $\frac{x}{1} - \frac{x^5}{5} + \frac{x^9}{9} - \&c = \frac{1}{4\sqrt{2}}\log_e\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$

5. $\frac{x}{1} - \frac{x^6}{6} + \frac{x^{11}}{11} - \&c = \frac{1}{10}\log_e\frac{(1+x)^5}{1+x^5}$

$+ \frac{1}{4\sqrt{5}}\log_e\frac{1+x\frac{\sqrt{5}-1}{2}+x^2}{1-x\frac{\sqrt{5}+1}{2}+x^2} + \frac{1}{10}\sqrt{10-2\sqrt{5}}\tan^{-1}\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)}$

$+ \frac{\sqrt{10+2\sqrt{5}}}{10}\tan^{-1}\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}-1)}$

6. $\frac{x}{1} - \frac{x^7}{7} + \frac{x^{13}}{13} - \frac{x^{19}}{19} + \&c = \frac{1}{2}\tan^{-1}x + \frac{1}{6}\tan^{-1}x^3$

$+ \frac{1}{4\sqrt{3}}\log_e\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}$

If $\Pi(a) = (1+a)(1+ax)(1+ax^2)$ &c . .

$$\frac{\Pi(a)\Pi(-b) - \Pi(-a)\Pi(b)}{\Pi(a)\Pi(-b) + \Pi(-a)\Pi(b)} = \frac{a-b}{1-x} + \frac{(a-bx)(ax-b)}{1-x^3} + \frac{x(a-bx^2)(ax^2-b)}{1-x^5} + \frac{x^2(a-bx^3)(ax^3-b)}{1-x^7} + \&c$$

$$\frac{(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})\&c}{(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})\&c} \cdot \frac{(1-b^2x^4)(1-b^2x^8)(1-b^2x^{12})\&c}{(1-b^2x^2)(1-b^2x^6)(1-b^2x^{10})\&c}$$

$$= \frac{1}{1-abx} + \frac{x(a-bx)(b-ax)}{(1+x^2)(1-abx)} + \frac{x(a-bx^3)(b-ax^3)}{(1+x^4)(1-abx)} +$$

$$\frac{(1-a^2x^3)(1-a^2x^7)(1-a^2x^{11})\&c}{(1-a^2x)(1-a^2x^5)(1-a^2x^9)} \cdot \frac{(1-b^2x^3)(1-b^2x^7)(1-b^2x^{11})\&c}{(1-b^2x)(1-b^2x^5)(1-b^2x^9)\&c}$$

$$= \frac{1}{1-ab} + \frac{(a-bx)(b-ax)}{(1+x^2)(1-ab)} + \frac{(a-bx^3)(b-ax^3)}{(1+x^4)(1-ab)} + \&c$$

7. $x - \frac{x^9}{9} + \frac{x^{17}}{17} - \frac{x^{25}}{25} + \&c =$

$$\frac{\sqrt{2+\sqrt2}}{16}\left\{\log_e \frac{1+x\sqrt{2+\sqrt2}+x^2}{1-x\sqrt{2+\sqrt2}+x^2} + 2\tan^{-1}\frac{x\sqrt{2+\sqrt2}}{1-x^2}\right\}$$

$$+\frac{\sqrt{2-\sqrt2}}{16}\left\{\log_e \frac{1+x\sqrt{2-\sqrt2}+x^2}{1-x\sqrt{2-\sqrt2}+x^2} + 2\tan^{-1}\frac{x\sqrt{2-\sqrt2}}{1-x^2}\right\}$$

8. $x - \frac{x^{11}}{11} + \frac{x^{21}}{21} - \frac{x^{31}}{31} + \&c = \frac{1}{4}\tan^{-1}x + \frac{1}{20}\tan^{-1}x^5$

$$+\frac{1}{4\sqrt5}\tan^{-1}\frac{(1+x^2)(1+x^2)\sqrt5\,x}{1-3x^2+x^4} + \frac{1}{40}\sqrt{10-2\sqrt5}\log_e\frac{1+\frac{x}{2}\sqrt{10-2\sqrt5}+x^2}{1-\frac{x}{2}\sqrt{10-2\sqrt5}+x^2}$$

$$+\frac{1}{40}\sqrt{10+2\sqrt5}\log_e\frac{1+\frac{x}{2}\sqrt{10+2\sqrt5}+x^2}{1-\frac{x}{2}\sqrt{10+2\sqrt5}+x^2}$$

(6). 1. $1 - \frac12 + \frac13 - \frac14 + \&c = \log_e 2$

2. $1 - \frac13 + \frac15 - \frac17 + \&c = \frac{\pi}{4}$

3. $1 - \frac14 + \frac17 - \frac{1}{10} + \&c = \frac{\pi}{3\sqrt3} + \frac13\log_e 2$

4. $1 - \frac15 + \frac19 - \frac{1}{13} + \&c = \frac{\pi}{4\sqrt2} + \frac{1}{2\sqrt2}\log_e(1+\sqrt2)$

5. $1 - \frac16 + \frac{1}{11} - \frac{1}{16} + \&c = \frac15\log_e 2 + \frac{1}{\sqrt5}\log_e\frac{1+\sqrt5}{2}$

$$+\frac{\pi}{50}\left(2\sqrt{10-2\sqrt5} + \frac{1}{2}\sqrt{10+2\sqrt5}\right)$$

6. $1 - \frac17 + \frac{1}{13} - \frac{1}{19} + \&c = \frac{\pi}{6} + \frac{1}{\sqrt3}\log_e\frac{1+\sqrt3}{\sqrt2}$

7. $1 - \frac19 + \frac{1}{17} - \frac{1}{25} + \&c = \frac{\pi}{8\sqrt2}\sqrt{2+\sqrt2}$

$$+\frac{1}{16}\sqrt{2+\sqrt2}\log_e\frac{2+\sqrt{2+\sqrt2}}{2-\sqrt{2+\sqrt2}} + \frac{1}{16}\sqrt{2-\sqrt2}\log_e\frac{2+\sqrt{2-\sqrt2}}{2-\sqrt{2-\sqrt2}}$$

8. $1 - \frac{1}{11} + \frac{1}{21} - \frac{1}{31} + \&c = \frac{\pi}{20} +$

$$\frac{\sqrt{10-2\sqrt5}}{40}\log_e\frac{4+\sqrt{10-2\sqrt5}}{4-\sqrt{10-2\sqrt5}} + \frac{\sqrt{10+2\sqrt5}}{40}\log_e\frac{4+\sqrt{10+2\sqrt5}}{4-\sqrt{10+2\sqrt5}}$$

9. $\frac{1}{1\cdot2} - \frac{1}{4\cdot5} + \frac{1}{7\cdot8} - \&c = \frac{\pi}{6\sqrt3} + \frac16\log_e 3$

10. $\frac{\sqrt3-1}{1} - \frac{(\sqrt3-1)^4}{4} + \frac{(\sqrt3-1)^7}{7} - \&c = \frac{\pi}{6\sqrt3} + \frac13\log_e\frac{1+\sqrt3}{\sqrt2}$

11. $\frac{2-\sqrt3}{1} - \frac{(2-\sqrt3)^3}{3} + \frac{(2-\sqrt3)^5}{5} - \&c = \frac{\pi}{16}(\sqrt3-1) - \frac{\sqrt3-1}{4}\log_e$ [illegible]

$$n - \frac{n}{\lfloor 1}(n+2)\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}$$

$$+ \frac{n(n+1)}{\lfloor 2}(n+4)\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times \frac{u(u-1)}{(u+n+1)(u+n+2)} - \&c$$

$$= n\cdot\frac{\lfloor x+n \; \lfloor y+n}{\lfloor n \; \lfloor x+y+z}\left\{1 + \frac{xy}{\lfloor 1}\cdot\frac{z+u+n+1}{(z+n+1)(u+n+1)} + \right.$$

$$\left.\frac{x(x-1)\,y(y-1)\,(z+u+n+1)(z+u+n+2)}{\lfloor 2\,(z+n+1)(z+n+2)(u+n+1)(u+n+2)} + \&c\right\}$$

$$\frac{\Pi(-a)\,\Pi(-d)}{\Pi(-b)\,\Pi(-c)}\left\{1 + \frac{a-b}{1-x}\cdot\frac{a-c}{a-d} + \frac{(a-b)(a-bx)(a-c)(a-cx)}{(1-x)(1-x^2)(a-d)(a-dx)} + \&c\right\}$$

$$= 1 + \frac{1-dx}{1-x}\cdot\frac{1-a}{a-d}\cdot\frac{b-d}{1-b}\cdot\frac{c-d}{1-c} + x^2\frac{(1-dx^3)(1-d)}{(1-x)(1-x^2)}\cdot\frac{(1-a)(1-ax)}{(a-d)(a-dx)}$$

$$\times \frac{(b-d)(b-dx)}{(1-b)(1-bx)}\cdot\frac{(c-d)(c-dx)}{(1-c)(1-cx)} + x^6\frac{(1-dx^5)(1-d)(1-dx)}{(1-x)(1-x^2)(1-x^3)}\cdot\frac{1-a}{a-d}$$

$$\times \frac{(1-ax)(1-ax^2)}{(a-dx)(a-dx^2)}\cdot\frac{(b-d)(b-dx)(b-dx^2)}{(1-b)(1-bx)(1-bx^2)}\cdot\frac{(c-d)(c-dx)(c-dx^2)}{(1-c)(1-cx)(1-cx^2)} + \&c$$

C. 1. $1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\&c = 2\log_e 2$

2. $1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\&c = \log_e 3$

3. $1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\&c = \frac{3}{2}\log_e 2$

4. $1+\frac{2}{5^3-5}+\frac{2}{10^3-10}+\frac{2}{15^3-15}+\&c = \frac{5}{4}\log_e 5 + \frac{\sqrt{5}}{4}\log_e\frac{\sqrt{5}+1}{2}$

5. $1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\frac{2}{18^3-18}+\&c = \frac{1}{2}\log_e 3 + \frac{2}{3}\log_e 4$

6. $1+\frac{2}{8^3-8}+\frac{2}{16^3-16}+\frac{2}{24^3-24}+\&c = \log_e 2 + \frac{\log_e(1+\sqrt{2})}{\sqrt{2}}$

7. $1+\frac{2}{10^3-10}+\frac{2}{20^3-20}+\frac{2}{30^3-30}+\&c = \frac{2}{5}\log_e 2 + \frac{1}{4}\log_e 5$

$+ \frac{3}{2\sqrt{5}}\log_e\left(\frac{1+\sqrt{5}}{2}\right)$

8. $1+\frac{2}{12^3-12}+\frac{2}{24^3-24}+\frac{2}{36^3-36}+\&c = \frac{1}{2}\log_e 2 + \frac{1}{2}\log_e 3$

$- \frac{1}{\sqrt{3}}\log_e(\sqrt{3}-1)$

9. $1+\frac{2}{16^3-16}+\frac{2}{32^3-32}+\frac{2}{48^3-48}+\&c = \frac{5}{8}\log_e 2 + \frac{1}{4\sqrt{2}}\log_e(1+\sqrt{2})$

$+ \frac{\sqrt{2+\sqrt{2}}}{16}\log_e\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} + \frac{\sqrt{2-\sqrt{2}}}{16}\log_e\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$

10. $1+\frac{2}{20^3-20}+\frac{2}{40^3-40}+\frac{2}{60^3-60}+\&c = \frac{1}{8}\log_e 5 + \frac{3}{10}\log_e 2$

$+ \frac{3}{4\sqrt{5}}\log_e\frac{\sqrt{5}+1}{2} + \frac{\sqrt{10-2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10-2\sqrt{5}}}{4-\sqrt{10-2\sqrt{5}}}$

$+ \frac{\sqrt{10+2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10+2\sqrt{5}}}{4-\sqrt{10+2\sqrt{5}}}$

15. If $\sum\frac{1}{x} = C_0 + \log_e a$, then

$\left(\frac{x+\frac{1}{2}}{a}\right)^{6n} = 1 - \frac{n}{1}\frac{1}{6a^2} + \frac{n(n+1\frac{1}{10})}{\lfloor 2}\frac{1}{(6a^2)^2} - \frac{n(n^2+3\frac{3}{10}n+12\frac{7}{10})}{(6a^2)^3\lfloor 3} + \&c$

Cor. $\lfloor x$ is minimum when $x = \frac{6}{13}$ very nearly.

Sol. $\lfloor x$ is maximum when ... i.e. $x=1$

$x - \ldots + \&c$ or $x =$ ... very nearly

$$\left\{1 + a\cdot\frac{1-b}{1-x}\cdot\frac{1-c}{1-d} + a^2\cdot\frac{(1-b)(1-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c\right\}$$

~~$\times (1-a)(1-ax)(1-ax^2)\ \&c$~~

$$\left\{1 + \frac{a-b}{1-x}\cdot\frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c\right\}$$

$$\times\frac{(1-a)(1-ax)(1-ax^2)\ \&c}{(1-b)(1-bx)(1-bx^2)\ \&c}$$

$$= 1 + \frac{a-b}{1-x}\cdot\frac{d-c}{1-d}\cdot\frac{1}{1-b}$$

$$+ \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(d-c)(dx-c)}{(1-d)(1-dx)}\cdot\frac{x}{(1-b)(1-bx)}$$

$$+ \frac{(a-b)(a-bx)(a-bx^2)}{(1-x)(1-x^2)(1-x^3)}\cdot\frac{(d-c)(dx-c)(dx^2-c)}{(1-d)(1-dx)(1-dx^2)}\cdot\frac{x^3}{(1-b)(1-bx)(1-bx^2)}$$

$$+ \&c$$

$$\frac{(1-ab)(1-abx)(1-abx^2)\ \&c}{(1-a)(1-ax)(1-ax^2)\ \&c}\cdot\frac{(1-ac)(1-acx)(1-acx^2)\ \&c}{(1-abc)(1-abcx)(1-abcx^2)\ \&c}$$

$$= 1 + a\cdot\frac{(1-b)(1-c)}{(1-a)(1-x)} + a^2\cdot\frac{(1-b)(x-b)\ (1-c)(x-c)}{(1-a)(1-ax)\ (1-x)(1-x^2)}$$

$$+ a^3\cdot\frac{(1-b)(x-b)(x^2-b)\ (1-c)(x-c)(x^2-c)}{(1-a)(1-ax)(1-ax^2)\ (1-x)(1-x^2)(1-x^3)} + \&c$$

16. $C_0 = \log_e 2 - 1\left(\frac{2}{3^2-3}\right) - 2\left(\frac{2}{6^2-6} + \frac{2}{9^2-9} + \frac{2}{12^2-12}\right) - \&c$

the next term in the latter group being $\frac{2}{(x^2+\frac{3}{2})^3 (x+\frac{1}{2})} =$

17. $\log_e \lfloor x = (x+\frac{1}{2})\log_e x - x + \frac{1}{2}\log_e(2\pi) + \frac{B_2}{1.2x} - \frac{B_4}{3.4x^3} + \&c$

Sol. Equate the coeffts of n in $\lfloor x$ 1.

The coefft of n in $\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$ = that of n in $-\frac{\lfloor n}{\pi(2\pi)^n} S_{n+1}\sin\frac{\pi n}{2}$

$\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$ = that of n in $-\frac{2}{2}(1 - n\log 2\pi + \&c)(\frac{1}{n} + C_0\ \&c)(1 - nC_0 + \&c)$

that of n in $-\frac{1}{2}\log_e(2\pi)$ =

= that of n in $-\frac{1}{2} + \frac{1}{2}\log_e(2\pi) + \&c$

or as follows:— let c be the constant in $\log_e \lfloor x$ &c. Let $f(x) = \log_e \frac{\lfloor 2x}{\lfloor x \lfloor x-\frac{1}{2}}$

then $f(x) - f(x-\frac{1}{2}) = \log_e 2$. ∴ $\log_e \frac{\lfloor 2x}{\lfloor x \lfloor x-\frac{1}{2}} = k + 2x\log_e 2$.

put x = 0. then we see that $k = \frac{1}{2}\log_e \pi$.

But the constant in $\log_e \frac{\lfloor 2x}{\lfloor x \lfloor x-\frac{1}{2}} = \frac{1}{2}\log_e 2 - c$.

∴ $c = \frac{1}{2}\log_e(2\pi) = .918938533204673$

Ex. Shew that when $x = \infty$ $\frac{e^x \lfloor x}{x^x\sqrt{2x+\frac{1}{3}}} = \sqrt{\pi}$.

18. $\log_e \lfloor x = -C_0 x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 - \&c$

i.e. $\log_e \frac{\lfloor x+2}{2} = .9227843351x + .197467033[illegible]x^2$
$- .0256856344x^3 + .0049558084x^4$
$- .0011355510x^5 + .0002863437x^6$
$- .0000[illegible]6825x^7 + .0000213843x^8$
$- .0000061409x^9 + .0000045404\frac{x^{10}}{3+x}$

Ex 1. $\log_e \lfloor -\frac{1}{2} = .5341910853$
$\log_e \lfloor -\frac{1}{3} = .1211436313$
$\log_e \lfloor -\frac{1}{4} = [illegible]7751287$

$$\frac{1}{2} + \frac{1}{1+(\frac{1}{x})^2} + \frac{1}{1+(\frac{2}{x})^2} + \cdots \cdots + \frac{1}{1+(\frac{x}{x})^2}$$

$$= \frac{\pi x}{2x}\coth \pi x - \frac{\pi x}{4x} - \frac{B_2}{2\times 1}\cdot\frac{1}{x} + \frac{B_6}{6\times 5}\frac{1}{x^3} - \frac{B_{10}}{10\times 9}\cdot\frac{1}{x^5} + \&c$$

$$\frac{(1-a)(1-ax)\&c\,(1-abc)(1-abcx)\&c\,(1-abd)(1-abdx)\&c\,(1-acd)\&c \times (1-acdx)\&c}{(1-ab)(1-abx)\&c\,(1-ac)(1-acx)\&c\,(1-ad)(1-adx)\&c \times (1-abcd)(1-abcdx)\&c}$$

$$= 1 - a\cdot\frac{(1-b)(1-c)(1-d)}{(1-ab)(1-ac)(1-ad)}\cdot\frac{1-ax}{1-x} + a^2\cdot\frac{(1-b)(x-b)(1-c)(x-c)}{(1-ab)(1-abx)(1-ac)(1-acx)}$$

$$\times\frac{(1-d)(x-d)}{(1-ad)(1-adx)}\cdot\frac{(1-ax^3)(1-a)}{(1-x)\,(1-x^2)} - a^3\cdot\frac{(1-b)(x-b)(x^2-b)}{(1-ab)(1-abx)(1-abx^2)}$$

$$\times\frac{(1-c)(x-c)(x^2-c)\,(1-d)(x-d)(x^2-d)}{(1-ac)(1-acx)(1-acx^2)(1-ad)(1-adx)(1-adx^2)}\cdot\frac{(1-ax^5)(1-a)(1-ax)}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \&c$$

19. $\Gamma(x)\,\Gamma(1-x) = \pi \operatorname{cosec} \pi x$. Cor. $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

20. $\Gamma(x)\,\Gamma(x+\tfrac{1}{n})\,\Gamma(x+\tfrac{2}{n}) \cdots \Gamma(x+\tfrac{n-1}{n}) = (2\pi)^{\frac{n-1}{2}} \, n^{\frac{1}{2}-nx}\,\Gamma(nx)$

Cor. $\Gamma(\tfrac{1}{n})\,\Gamma(\tfrac{2}{n})\,\Gamma(\tfrac{3}{n}) \cdots \Gamma(\tfrac{n-1}{n}) = \dfrac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$

Ex 1. $(\Gamma(\tfrac{1}{4}))^2 = \ldots \sqrt{\pi} \ldots$

2. $\Gamma(\tfrac{1}{6}) \ldots \Gamma(\tfrac{5}{6}) = 2\pi \ldots$

3. $2^{2x}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) = \ldots \sqrt{\pi}\,\Gamma(2x)$

4. $\dfrac{\Gamma(2x)}{\Gamma(x)\,\Gamma(x+\tfrac{1}{2})} = \dfrac{2^{2x-1}}{\sqrt{\pi}}$

5. $\log_e \Gamma(x+1) = x \log_e x - x + \log_e \sqrt{2\pi} + \ldots \dfrac{B_2}{1.2x} - \ldots$

27. $\dfrac{\log 1}{1} + \dfrac{\log 2}{2} + \cdots + \dfrac{\log x}{x} = \tfrac{1}{2}(\log_e x)^2 + \phi(x)$

$\phi(x) = (\tfrac{1}{2x} - c_1)\log_e x - \ldots + \dfrac{B_2}{2x^2}\,1 - \dfrac{B_4}{4x^4}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \dfrac{B_6}{6x^6}(1+ \ldots) - \&c$

$c_1 = -\cdot 0728 \ldots 680$

Sol. Write $n+1$ for x in 1 then divide both sides by x^n and find the coefficients of x ... both sides ...

Ex when $x = \infty$ show that $\phi(x) \ldots = -c_1$

31. $\phi(x) - \dfrac{\log 1}{1} - \dfrac{\log(1+x)}{1+x} + \dfrac{\log 2}{2} - \dfrac{\log(2+x)}{2+x} \ldots$

Cor. $\dfrac{\log 1}{1} - \dfrac{\log 2}{2} + \dfrac{\log 3}{3} - \ldots \&c = \ldots \log 2 + \tfrac{1}{2}\{\phi(\ldots) - \phi(\ldots)\}$

33. $n\phi(x) - \{\phi(\tfrac{x}{n}) + \phi(\tfrac{x+1}{n}) + \cdots + \phi(\tfrac{x+n-1}{n})\}$

$= \ldots \log_e n \ldots - \tfrac{1}{2}(\log n)^2$

$$\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\cdots = \phi(x)$$

$$\phi\left(\frac{x}{1-y}\right)+\phi\left(\frac{y}{1-x}\right)$$
$$=\phi(x)+\phi(y)+\phi\left\{\frac{xy}{(1-x)(1-y)}\right\}+\log(1-x)\log(1-y).$$

$$\frac{1+\frac{a-b}{1-x}\cdot\frac{1-c}{1-d}+\frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)}+\&c}{1+\frac{c-d}{1-x}\cdot\frac{1-a}{1-b}+\frac{(c-d)(c-dx)}{(1-x)(1-x^2)}\frac{(1-a)(1-ax)}{(1-b)(1-bx)}+\&c}$$

$$=\frac{(1-b)(1-bx)(1-bx^2)\ \&c}{(1-a)(1-ax)(1-ax^2)\ \&c}\cdot\frac{(1-c)(1-cx)(1-cx^2)\ \&c}{(1-d)(1-dx)(1-dx^2)\ \&c}$$

$$\int_0^\infty\left\{\frac{n^x\lfloor n}{\lfloor n+x}+e^{-x}\left(1+\frac{x}{n}\right)^n\right\}dx=\frac{e^n\lfloor n}{n^n}+\frac{6n}{12n+1}$$

very very nearly.

$$1+\frac{n}{\lfloor 1}+\frac{n^2}{\lfloor 2}+\cdots+\frac{n^n}{\lfloor n}\theta=\frac{e^n}{2}.$$

where $\theta=\frac{4+15n}{8+45n}$ very nearly.

or still ~~more approximately~~

$\theta=\frac{2+7nu}{4+21nu}$ · ~~where u~~ $=\frac{466+585n}{480+546n}$.

Cor. $\phi(-\frac{1}{n}) + \phi(-\frac{2}{n}) + \phi(-\frac{3}{n}) + \cdots + \phi(-\frac{n-1}{n})$

$= nC_0 \log_e n + \frac{n}{2}(\log_e n)^2$

Ex. 1. $\frac{\sqrt[1]{1}}{\sqrt[2]{2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[4]{4}} \cdot \frac{\sqrt[5]{5}}{\sqrt[6]{6}} \cdot \frac{\sqrt[7]{7}}{\sqrt[8]{8}} \cdot \frac{\sqrt[9]{9}}{\sqrt[10]{10}} \cdots$ ad inf.

2. $\phi(-\frac{1}{2}) = (\log_e 2)^2 + 2C_0 \log_e 2$

3. $\phi(-\frac{1}{3}) + \phi(-\frac{2}{3}) = \frac{3}{2}(\log_e 3)^2 + 3C_0 \log_e 3.$

4. $\phi(-\frac{1}{4}) + \phi(-\frac{3}{4}) = 7(\log_e 2)^2 + 6C_0 \log_e 2.$

5. $\phi(-\frac{1}{6}) + \phi(-\frac{5}{6}) = C_0(3\log_e 3 + 4\log_e 2)$ [illegible]

24. $\frac{\pi}{2}\left\{\log_e \frac{\lfloor x-1}{\lfloor -x} + (C_0 + \log_e 2\pi)(2x-1)\right\}$ when x lies between 0 & 1.

$= \frac{\log 1}{1}\sin 2\pi x + \frac{\log 2}{2}\sin 4\pi x + \frac{\log 3}{3}\sin 6\pi x + \&c$

N.B. $\frac{\pi}{2} - \pi x = \sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \&c$ for the same limits

25. $\phi(x-1) - \phi(-x) = (C_0 + \log_e 2\pi)\,\pi \cot \pi x$

$+ 2\pi\left\{\sin 2\pi x \frac{\log_e 1}{1} + \sin 4\pi x \frac{\log 2}{2} + \&c\right\}$

N.B. $\sin 2\pi x + \sin 4\pi x + \sin 6\pi x + \&c = \frac{1}{2}\cot \pi x.$

Ex. 1. Find the values of $\phi(-\frac{1}{4})$, $\phi(-\frac{1}{3})$, $\phi(-\frac{2}{3})$ & $\phi(-\frac{5}{6})$.

2. $\frac{\log 1}{1} - \frac{\log 3}{3} + \frac{\log 5}{5} - \frac{\log 7}{7} + \&c = \frac{\pi}{4}\log_e \pi - \pi \log_e \lfloor -\frac{1}{4} + \frac{\pi}{4}C_0.$

3. $\frac{\left(\frac{\sqrt[1]{1}}{\sqrt[2]{2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[4]{4}} \cdot \frac{\sqrt[5]{5}}{\sqrt[6]{6}} \cdot \frac{\sqrt[7]{7}}{\sqrt[8]{8}} \cdots\right)^{\frac{1}{\log_e 2}}}{\left(\frac{\sqrt[1]{1}}{\sqrt[3]{3}} \cdot \frac{\sqrt[5]{5}}{\sqrt[7]{7}} \cdot \frac{\sqrt[9]{9}}{\sqrt[11]{11}} \cdot \frac{\sqrt[13]{13}}{\sqrt[15]{15}} \cdots\right)^{\frac{4}{\pi}}} = \frac{\sqrt{2}}{\pi}\left(\lfloor -\frac{3}{4}\right)^4$

26. $(\log_e 1)^2 + (\log_e 2)^2 + (\log_e 3)^2 + \cdots + (\log_e x)^2 = \psi(x)$

$\psi(x) = 2\log_e 2 \log_e \frac{\lfloor x}{\sqrt{2\pi}} - (x+\frac{1}{2})(\log_e x)^2 + 2x + \frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24}$ [illegible]

$+ 2\left\{\frac{B_2}{1 \cdot 2} \cdot \frac{1+\frac{1}{2}}{x} - \frac{B_4}{3 \cdot 4} \cdot \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{x^3}\right.$ [illegible]

Sol. Equate the coeff. of [illegible]

$$* \quad 1 + \frac{\beta x}{(1-x)(1-\alpha x)} + \frac{\beta^2 x^3}{(1-x)(1-x^2)(1-\alpha x)(1-\alpha x^2)} + \&c$$

$$\times (1-\alpha x)(1-\alpha x^2)(1-\alpha x^3)\ \&c$$

$$= 1 - x \cdot \frac{\alpha-\beta}{1-x} + x^3 \cdot \frac{(\alpha-\beta)(\alpha-\beta x)}{(1-x)(1-x^2)}$$

$$- x^6 \cdot \frac{(\alpha-\beta)(\alpha-\beta x)(\alpha-\beta x^2)}{(1-x)(1-x^2)(1-x^3)} + \&c \quad \sout{= F(\beta)}$$

$$\frac{F(\beta)}{F(\beta x)} = 1 + \cfrac{\beta x}{1-\alpha x + \cfrac{\beta x^2}{1-\alpha x^2 + \cfrac{\beta x^3}{1-\alpha x^3 + \&c}}}$$

where $* = F(\beta)$.

$$\cfrac{1}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+ \cdots + \cfrac{a_{n+2}}{1}}}}} = \frac{N}{D}.$$

$$D = \phi_0(n) + \phi_1(n) + \phi_2(n) + \&c.$$

& N is the D of $\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+ \cdots \cfrac{a_{n+2}}{1}}}}$

27. $\psi(x) - \left\{\psi\left(\frac{x}{n}\right) + \psi\left(\frac{x-1}{n}\right) + \psi\left(\frac{x-2}{n}\right) + \cdots + \psi\left(\frac{x-n+1}{n}\right)\right\}$

$= 2\log_e n \log_e \frac{\Gamma(x+1)}{\sqrt{2\pi}} - x(\log_e n)^2 - (n-1)\left\{\frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}(\log_e 2\pi)^2\right\} - \frac{1}{2}(\log_e n)^2$

Cor. $\psi\left(-\frac{1}{n}\right) + \psi\left(-\frac{2}{n}\right) + \cdots + \psi\left(-\frac{n-1}{n}\right)$

$= \log_e n \log_e 2\pi + (n-1)\left\{\frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}(\log_e 2\pi)^2\right\} + \frac{1}{2}(\log_e n)^2$

Ex. 1. $\frac{1^{\frac{1}{1}} \cdot 2^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} \cdot 4^{\frac{1}{4}} \cdots x^{\frac{1}{x}}}{[illegible]}$ [illegible]

$= e^{\frac{\pi^2}{24}}(2\pi)^{\frac{1}{2}\log 2\pi}$ when $x = \infty$. [illegible]

2. $\psi\left(-\frac{1}{2}\right) = [illegible] \frac{\pi^2}{24} - \frac{1}{2}\left\{(\log_e 2)^2 + (\log_e \pi)^2\right\} + (\log_e 2)^2$

3. $\psi\left(-\frac{1}{3}\right) + \psi\left(-\frac{2}{3}\right) = C_0^2 + 2C_1 - \frac{\pi^2}{12} - (\log_e 2\pi)^2 + \log_e 3 \log_e 2\pi + \frac{1}{2}(\log_e 3)^2$

4. $\psi\left(-\frac{1}{4}\right) + \psi\left(-\frac{3}{4}\right) = C_0^2 + 2C_1 - \frac{\pi^2}{12} - \log_e \pi \log_e 2\pi + 1\frac{1}{2}(\log_e 2)^2$

5. $\frac{1}{2}\left\{\psi\left(-\frac{1}{6}\right) + \psi\left(-\frac{5}{6}\right)\right\} = \frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}(\log_e 2\pi)^2 + \frac{1}{2}\log_e 2 \log_e 3.$

28. $\frac{\psi(x-1) + \psi(-x)}{2} = C_1 - \frac{\pi^2}{24} + \frac{1}{2}(C_0 + \log_e 2\pi)\left(C_0 - \log_e \frac{\pi}{2\sin \pi x}\right)$

[illegible] $\left\{\frac{\log 2}{1}\cos 2\pi x + \frac{\log 2}{2}\cos 4\pi x + \&c\right\}$

29. If C_n be the constant in $(\log_e 1)^n + (\log_e 2)^n + \cdots (\log_e x)^n$ and if $\phi_n(x) = (\log_e 1)^n + (\log_e 2)^n + \cdots + (\log_e x)^n - C_n$, then

i. The logarithmic part of $\phi_n(x) = n \log_e x\, \phi_{n-1}(x) - \frac{n(n-1)}{\lfloor 2}(\log_e x)^2 \phi_{n-2}(x) + \frac{n(n-1)(n-2)}{\lfloor 3}(\log_e x)^3 \phi_{n-3}(x)$ &c. and the non logarithmic part can be found from 23.

ii. $\phi_0(x)(\log_e x)^n - \frac{n}{\lfloor 1}\phi_1(x)(\log_e x)^{n-1} + \frac{n(n-1)}{\lfloor 2}\phi_2(x)(\log_e x)^{n-2}$ &c

$= x\lfloor n - \frac{1}{x^n}\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2} - \frac{n}{2}\cdot\frac{1}{x^{n+1}}\frac{B_{n+2}}{n+2}\cos\frac{\pi n}{2}$

$+ \frac{n(n+1)}{2 \cdot 3}\cdot\frac{1}{x^{n+2}}\cdot\frac{B_{n+3}}{n+3}\sin\frac{\pi n}{2} + \frac{n(n+1)(n+2)}{2 \cdot 3 \cdot 4 \cdots}\cdots\frac{B_{n+4}}{n+4}$

$\times \cos\frac{\pi n}{2}$

$$1+\frac{x^2}{(1-x)^2}+\frac{x^6}{(1-x)^2(1-x^2)^2}+\frac{x^{12}}{(1-x)^2(1-x^2)^2(1-x^3)^2}+\&c$$

$$=\frac{1-x+x^3-x^6+x^{10}-\&c}{(1-x)(1-x^2)(1-x^3)(1-x^4)\&c}$$

$$\frac{(1+xy)(1+x^3y)(1+x^5y)\&c}{(1+\alpha xy)(1+\alpha x^3y)(1+\alpha x^5y)\&c}\cdot\frac{(1+\frac{x}{y})(1+\frac{x^3}{y})(1+\frac{x^5}{y})\&c}{(1+\beta\frac{x}{y})(1+\beta\frac{x^3}{y})(1+\beta\frac{x^5}{y})\&c}$$

$$\times\frac{(1-x^2)(1-x^4)(1-x^6)\&c\cdot(1-\alpha\beta x^2)(1-\alpha\beta x^4)(1-\alpha\beta x^6)\&c}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)\&c\cdot(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)\&c}$$

$$=1+\left\{xy\cdot\frac{1-\alpha}{1-\beta x^2}+\frac{x}{y}\cdot\frac{1-\beta}{1-\alpha x^2}\right\}$$

$$+\left\{(xy)^2\cdot\frac{(1-\alpha)(x^2-\alpha)}{(1-\beta x^2)(1-\beta x^4)}+\left(\frac{x}{y}\right)^2\frac{(1-\beta)(x^2-\beta)}{(1-\alpha x^2)(1-\alpha x^4)}\right\}$$

$$+\left\{(xy)^3\cdot\frac{(1-\alpha)(x^2-\alpha)(x^4-\alpha)}{(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)}+\left(\frac{x}{y}\right)^3\frac{(1-\beta)(x^2-\beta)(x^4-\beta)}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)}\right\}$$

$+\ \&c\ \ \&c\ \ \&c$

$$- \frac{n(n+2)(n+4)^2 + \frac{n(n+2)}{3} + \frac{4n}{5}}{2 \cdot 4 \cdot 6 \cdot 8} \frac{1}{x^{n+4}} \frac{B_{n+5}}{n+5} \sin\frac{\pi n}{2}$$

$$- \frac{n(n+4)(n+5)\left\{(n+3)(n+4) + \frac{1}{3}(n+1)\right\}}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \frac{1}{x^{n+5}} \frac{B_{n+6}}{n+6} \cos\frac{\pi n}{2} + \&c.$$

iii. $\phi_n\left(\frac{x}{a}\right) + \phi_n\left(\frac{x-1}{a}\right) + \cdots + \phi_n\left(\frac{x-a+1}{a}\right)$

$= \phi_n(x) - n \log_e a \, \phi_{n-1}(x) + \frac{n(n-1)}{\lfloor 2}(\log_e a)^2 \phi_{n-2}(x) - \&c.$

Cor 1. $\phi_n\left(-\frac{1}{a}\right) + \phi_n\left(-\frac{2}{a}\right) + \cdots + \phi_n\left(-\frac{a-1}{a}\right)$

$= -\left\{C_n - n \log_e a \, C_{n-1} + \frac{n(n-1)}{\lfloor 2}(\log_e a)^2 C_{n-2} - \&c\right\}$

Cor 2. There will be no logarithmic functions in

$\phi_n\left(\frac{1}{x}\right) + \phi_n\left(\frac{2}{x}\right) + \cdots + \phi_n\left(\frac{x}{x}\right)$

30. Let $1^a + 2^a k + 3^a k^2 + \cdots + x^a k^{x-1} = k^x \phi(x) = F_k(x)$, then

i. $\phi(x) = C_a(k) + x^a \frac{\psi_0(-k)}{k-1} - \frac{a}{\lfloor 1} x^{a-1} \frac{\psi_1(-k)}{(k-1)^2} +$

$\frac{a(a-1)}{\lfloor 2} \frac{\psi_2(k)}{(k-1)^3} - \&c$ where ψ [illegible] the same [illegible]

ii. $C_a(k) = \frac{\psi_a(-k)}{(1-k)^{a+1}}$ &c $k^a \psi_a\left(-\frac{1}{k}\right) = k^a \psi\left(-\frac{1}{k}\right)$.

iii. $F_k\left(\frac{x}{n}\right) + F_k\left(\frac{x-1}{n}\right) + \cdots + F_k\left(\frac{x-n+1}{n}\right) - n C_a(k)$

$= \frac{\sqrt[n]{k}}{k n^a}\left\{F_{\sqrt[n]{k}}(x) - C_a(\sqrt[n]{k})\right\}$

Cor. $F_k\left(-\frac{1}{n}\right) + F_k\left(-\frac{2}{n}\right) + \cdots + F_k\left(-\frac{n-1}{n}\right) = n C_a(k) - \frac{\sqrt[n]{k}\, C_a(\sqrt[n]{k})}{k n^a}$

31. Let $\frac{\log_e 1}{1^n} + \frac{\log_e 2}{2^n} + \frac{\log_e 3}{3^n} + \cdots + \frac{\log_e x}{x^n} = \phi_n(x)$ & let C_n be the constant. Then

i. $\phi_n(x) = C_n - \left\{\frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \frac{1}{(x+3)^n} + \&c\right\} \log_e x - \frac{1}{(n-1)^2 x^{n-1}}$

$+ B_2 \frac{n}{\lfloor 2} \frac{1}{n x^{n+1}} - B_4 \frac{n(n+1)(n+2)}{\lfloor 4}\left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}\right)\frac{1}{x^{n+3}}$

$+ B_6 \frac{n(n+1)(n+2)(n+3)(n+4)}{\lfloor 6}\left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4}\right)\frac{1}{x^{n+5}}$ &c

If $\displaystyle\int_{\alpha_1}^{\beta_1} \phi_1(p, x)\, F(nx)\, dx = \psi_1(p, n)$

& $\displaystyle\int_{\alpha_2}^{\beta_2} \phi_2(p, x)\, F(nx)\, dx = \psi_2(p, n)$

then $\displaystyle\int_{\alpha_1}^{\beta_1} \phi_1(p, x)\, \psi_2(q, nx)\, dx = \int_{\alpha_2}^{\beta_2} \phi_2(q, x)\, \psi_1(p, x)$

$$\int_{-\infty}^{\infty} \frac{e^{mx} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(px)}{(1+e^x)^n}\, dx$$

$$= \frac{\underline{|m-1}\ \underline{|n-m-1}}{\underline{|n-1}} \cdot \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left(\tan^{-1}\frac{p}{n-m} - \tan^{-1}\frac{p}{m} + \tan^{-1}\frac{p}{n-m+1} - \tan^{-1}\frac{p}{m+1} + \&c\right)$$

$$\div \sqrt{\left\{1 + \frac{p^2}{(n-m)^2}\right\}\left\{1 + \frac{p^2}{m^2}\right\}\left\{1 + \frac{p^2}{(n-m+1)^2}\right\}\left\{1 + \frac{p^2}{(m+1)^2}\right\}\&c}$$

$$\int_0^{\infty} e^{-mx}(1-e^{-x})^n \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(px)\, dx$$

ii. $\phi_r(x) = rx\,C'_{r+1} - \frac{r(r+1)}{\lfloor 2}x^2 C'_{r+2} + \frac{r(r+1)(r+2)}{\lfloor 3}x^3 C'_{r+3} - \&c$

$- r\cdot\frac{1}{r}\,x\,S_{r+1} + \frac{r(r+1)}{\lfloor 2}\left(\frac{1}{r}+\frac{1}{r+1}\right)x^2 S_{r+2} - \&c.$

iii. $n^r\phi_r(x) - \left\{\phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \cdots\cdots + \phi_r\left(\frac{x-n+1}{n}\right)\right\}$

$= C'_r(n^r - n) - n^r\log_e n\left\{\frac{1}{(x+1)^r} + \frac{1}{(x+2)^r} + \frac{1}{(x+3)^r} + \&c\right\}$

Cor. $\phi_r\left(-\frac{1}{n}\right) + \phi_r\left(-\frac{2}{n}\right) + \phi_r\left(-\frac{3}{n}\right) + \cdots\cdots + \phi_r\left(-\frac{n-1}{n}\right)$

$= n^r\log_e n\,S_r - (n^r - n)\,C'_r$

32. Let $(\log_e 1)^r + \frac{1}{2}(\log_e 2)^r + \frac{1}{3}(\log_e 3)^r + \&c$ to x terms $= \phi_r(x)$ & let C_r be its constant. then

i. $\phi_r(x) = \frac{1}{r+1}(\log_e x)^{r+1} + C_r$ when $x = \infty$

ii. $n\,\phi_r(x) - \left\{\phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \phi_r\left(\frac{x-2}{n}\right) + \cdots + \phi_r\left(\frac{x-n+1}{n}\right)\right\}$

$= \frac{n}{r+1}(\log_e n)^{r+1}\cos\pi r + rn\log_e n\left\{\phi_{r-1}(x) - C_{r-1}\right\}$

$\frac{r(r-1)}{\lfloor 2}\,n(\log_e n)^2\left\{\phi_{r-2}(x) - C_{r-2}\right\} + \&c$ the last term being

$(-1)^{r-1}n(\log_e n)^r\left\{\phi_0(x) - C_0\right\}$

33. $\frac{(\log_e 1)^r}{1^{n+1}} + \frac{(\log_e 2)^r}{2^{n+1}} + \frac{(\log_e 3)^r}{3^{n+1}} + \&c$

$= \frac{\lfloor r}{n^{r+1}} + C_r - \frac{n}{\lfloor 1}C_{r+1} + \frac{n^2}{\lfloor 2}C_{r+2} - \frac{n^3}{\lfloor 3}C_{r+3} + \&c$

Sol. Differentiate r times both the sides.

Ex. Show that

1. $\frac{(\log_e 1)^3}{1\sqrt{1}} + \frac{(\log_e 2)^3}{2\sqrt{2}} + \frac{(\log_e 3)^3}{3\sqrt{3}} + \&c = 96.001$ nearly.

2. $\frac{\log_e 1}{1^2} + \frac{\log_e 2}{2^2} + \frac{\log_e 3}{3^2} + \&c = .9382$ nearly.

3. $\frac{(\log_e 1)^4}{1^2} + \frac{(\log_e 2)^4}{2^2} + \frac{(\log_e 3)^4}{3^2} + \&c = 24$ nearly

4. $\frac{(\log_e 1)^5}{1\sqrt{1}} + \frac{(\log_e 2)^5}{2\sqrt{2}} + \frac{(\log_e 3)^5}{3\sqrt{3}} + \&c = 7680$ nearly.

If $\int_0^\infty \phi_1(x) F(nx)\,dx = \psi_1(n)$

& $\int_0^\infty \phi_2(x) F(nx)\,dx = \psi_2(n)$

then $\int_0^\infty \phi_1(ax)\psi_2(bx)\,dx = \int_0^\infty \phi_2(ax)\psi_1(bx)$

If $\left.\begin{array}{l}\int_{\alpha_1}^{\beta_1} \phi_1(x) F(nx)\,dx = \psi_1(n) \\ \& \int_{\alpha_2}^{\beta_2} \phi_2(x) F_1(nx)\,dx = \psi_2(n)\end{array}\right\}$ then

$$\int_{\alpha_1}^{\beta_1} \phi_1(x)\,\psi_2(nx)\,dx = \int_{\alpha_2}^{\beta_2} \phi_2(x)\,\psi_1(nx)\,dx$$

5. $\frac{(\log 1)^5}{1^2}\sqrt{\log 1} + \frac{(\log 2)^5}{2^2}\sqrt{\log 2} + \frac{(\log 3)^5}{3^2}\sqrt{\log 3} + \&c = $ [illegible] nearly.

14. $\frac{\log 1}{\sqrt{1}} + \frac{\log 2}{\sqrt{2}} + \frac{\log 3}{\sqrt{3}} + \cdots + \frac{\log x}{\sqrt{x}} = \phi(x)$

i. $\phi(x) = \frac{\log 1}{\sqrt{1}} - \frac{\log(1+x)}{\sqrt{1+x}} + \frac{\log 2}{\sqrt{2}} - \frac{\log(2+x)}{\sqrt{2+x}} + \&c$

ii. $\phi(x) = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}\right)\log_e x$

$+ (\sqrt{2}+1)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left(\log_e x - \frac{1}{2}C_0 + \frac{\pi}{4} + \log_e \sqrt{8\pi}\right)$ [illegible]

$- 4\sqrt{x} + \frac{1}{2}\cdot\frac{B_2}{x\sqrt{x}} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_4}{2x^3\sqrt{x}}$

$+ \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)\frac{B_6}{3x^5\sqrt{x}}$ [illegible]

iii. $\phi(x) - \frac{1}{\sqrt{n}}\left\{\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)\right\}$

$= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{x}}\right)\log_e n$

$- (1+\sqrt{2})\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left\{(\sqrt{n}-1)\right.$ [illegible]

iv. If $\psi(x) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}$, then

$\{\phi(x-1) + \phi(-x) - 2C\} + \left(C_0 + \frac{\pi}{2} + \log 8\pi\right)\{\psi(x-1) + \psi(-x) - 2C'\}$

$= 2\left\{\frac{\log 1}{\sqrt{1}}\sin 2\pi x + \frac{\log 2}{\sqrt{2}}\cos 4\pi x + \&c\right\}$

v. $\{\phi(x-1) - \phi(-x)\} + \left(C_0 - \frac{\pi}{2} + \log 8\pi\right)\{\psi(x-1) - \psi(-x)\}$

$= 2\left\{\frac{\log 1}{\sqrt{1}}\sin 2\pi x + \frac{\log 2}{\sqrt{2}}\sin 4\pi x + \&c\right\}$

In both cases C & C' are the constants of φ(x) & ψ(x) respectively.

Ex 1. Find the values of $\phi(-\frac{1}{2})$, $\phi(-\frac{3}{2})$ [illegible]

2. Show that the constant in $\phi(x) = -\frac{\phi}{2}\left(C_0 + \frac{\pi}{2} + \log 8\pi\right)$

$= 3.92265\ldots = 2\left\{1 - \frac{1}{2}\cdot B_2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_4}{2} - \&c\right\}$

Sol. Write ½ for x in IV & [illegible] the constant [illegible] in I & II; then the end result is [illegible] obtained.

$\phi(a) + \phi(a) + \cdots + \phi(h)$

$= \int^{h} \phi(x)\,dx + \frac{1}{2}\phi(h) + \int_0^\infty \frac{\phi(h+xi) - \phi(h-xi)}{i(e^{2\pi x}-1)}\,dx$

The imaginary part in

$\phi(n)\,F(a+bi) - \phi(2n)\,F(a+2bi) + \phi(3n)\,F(a+3bi) - \&c$

$$= \int_0^\infty \frac{F(a+bx) - F(a-bx)}{e^{\pi x} - e^{-\pi x}}\,\phi(n^2x^2)\,dx.$$

$$\int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cdot \frac{dx}{1+n^2x^2}$$

$$= \frac{\sin a}{1+n} - \frac{\sin 2a}{1+2n} + \frac{\sin 3a}{1+3n} - \&c$$

a lying between 0 & π.

$$\operatorname{cosec}\theta + h\left\{\frac{\sin\theta}{e^{y}-1} + \frac{\sin 3\theta}{e^{3y}-1} + \frac{\sin 5\theta}{e^{5y}-1}\right\}$$

$= z \operatorname{cosec}\phi$

$$\frac{\cos\theta}{\sinh\frac{y}{2}} - \frac{\cos 3\theta}{\sinh\frac{3y}{2}} + \frac{\cos 5\theta}{\sinh\frac{5y}{2}} - \&c$$

$$= \frac{z}{2}\sqrt{x}\,.\,\frac{\cos\phi}{\sqrt{1 - x\sin^2\phi}}$$

1. If $S_n = \frac{1}{(1-a)^n} - \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} - \frac{1}{(3+a)^n} + \&c$, then

i. If n is odd,

$$\frac{\cos(1-a)x}{(1-a)^n} - \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} - \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{\lfloor 2} S_{n-2} + \frac{x^4}{\lfloor 4} S_{n-4} - \&c$ as far as the term containing [illegible]

ii. If n is even

$$\frac{\sin(1-a)x}{(1-a)^n} - \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} - \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{\lfloor 1} S_{n-1} - \frac{x^3}{\lfloor 3} S_{n-3} + \frac{x^5}{\lfloor 5} S_{n-5} - \&c$ as far as the term containing S_1

2. If $S_n = \frac{1}{(1-a)^n} + \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} + \frac{1}{(3+a)^n} + \&c$, then

i. If n is even

$$\frac{\cos(1-a)x}{(1-a)^n} + \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} + \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{\lfloor 2} S_{n-2} + \frac{x^4}{\lfloor 4} S_{n-4} - \&c$ as far as the term containing S_2

ii. If n is odd

$$-\frac{\sin(1-a)x}{(1-a)^n} + \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} + \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{\lfloor 1} S_{n-1} - \frac{x^3}{\lfloor 3} S_{n-3} + \&c$ as far as the term containing S_2

[illegible] expand the series in ascending powers of x and apply

1. If $\phi(x) = \frac{\cos x}{1} - (1+\frac{1}{2})\frac{\cos 2x}{2} + (1+\frac{1}{2}+\frac{1}{3})\frac{\cos 3x}{3} - \&c$ [illegible]

$$\int_0^{\infty} e^{-x}\frac{x^p}{\lfloor p}\left\{\int_0^{\infty}\frac{e^{-y}}{x+y}\cdot\frac{y^{p+q-1}}{\lfloor p+q-1}\,dy\right\}^n dx$$

$$= \frac{1}{q^n + p\left(\frac{\lfloor q-1}{\lfloor q-n} - q^{n-1}\right)}$$

$$\frac{1}{\sin^2\theta} - 8\left(\frac{1\cos 2\theta}{e^{2y}-1} + \frac{2\cos 4\theta}{e^{4y}-1} + \frac{3\cos 6\theta}{e^{6y}-1} + \&c\right)$$

$$= \frac{z^2}{\sin^2\phi} - z^2\left(\frac{1+x}{3}\right) + \frac{1}{3}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$\frac{\cos\theta}{\sin^3\theta} - 8\left(\frac{1^2\sin 2\theta}{e^{2y}-1} + \frac{2^2\sin 4\theta}{e^{4y}-1} + \&c\right)$$

$$= z^3\frac{\cos\phi}{\sin^3\phi}\sqrt{1-x\sin^2\phi}$$

$$\text{when } \theta z = \int_0^{\phi}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}.$$

$$\sec\theta + 4\left\{\frac{\cos\theta}{e^{y}-1} - \frac{\cos 3\theta}{e^{3y}-1} + \frac{\cos 5\theta}{e^{5y}-1} - \&c\right\}$$

$$= z\sec\phi\sqrt{1-x\sin^2\phi}$$

$$\frac{\sin\theta}{\cosh\frac{y}{2}} - \frac{\sin 3\theta}{\cosh\frac{3y}{2}} + \frac{\sin 5\theta}{\cosh\frac{5y}{2}} - \&c =$$

$$= \frac{z}{2}\sqrt{x(1-x)}\cdot\frac{\sin\phi}{\sqrt{1-x\sin^2\phi}}$$

If n is odd $\phi(n-2) + \phi(n) =$

$$x\left\{\left(\frac{\sin x}{1^{n-2}} - \frac{\sin 3x}{3^{n-2}} + \frac{\sin 5x}{5^{n-2}} - \&c\right) - \left(\frac{\sin x}{1^n} - \frac{\sin 3x}{3^n} + \frac{\sin 5x}{5^n} - \&c\right)\right\}$$

$$+ n\left\{\left(\frac{\cos x}{1^{n-1}} - \frac{\cos 3x}{3^{n-1}} + \frac{\cos 5x}{5^{n-1}} - \&c\right) - \left(\frac{\cos x}{1^{n+1}} - \frac{\cos 3x}{3^{n+1}} + \frac{\cos 5x}{5^{n+1}} - \&c\right)\right\}$$

4\. i. $\frac{1}{2}\sin 2x - \frac{1+\frac{1}{2}}{3}\sin 3x + \frac{1+\frac{1}{2}+\frac{1}{3}}{4}\sin 4x - \&c$

$= \frac{x}{2}\left(\frac{\cos x}{1} - \frac{\cos 2x}{2} + \frac{\cos 3x}{3} - \&c\right)$

ii. $\cos 2x - (1+\frac{1}{2})\cos 4x + (1+\frac{1}{2}+\frac{1}{3})\cos 6x - \&c$

$= \frac{x}{2}(\cos x - \cos 3x + \cos 5x - \&c)$ [illegible]

Integrate both sides [illegible]

5\. i. $\sin a\theta + \frac{n}{\lfloor 1}\sin(a+2)\theta + \frac{n(n-1)}{\lfloor 2}\sin(a+4)\theta + \&c =$

$= (2\cos\theta)^n \sin(a+n)\theta$

ii. $\cos a\theta + \frac{n}{\lfloor 1}\cos(a+2)\theta + \frac{n(n-1)}{\lfloor 2}\cos(a+4)\theta + \&c$

$= (2\cos\theta)^n \cos(a+n)\theta$

Sol. find $e^{a\theta i} + \frac{n}{\lfloor 1} e^{(a+2)\theta i} + \ldots$ [illegible]

and apply De Moivre's theorem

6\. i. $\sin x - \frac{1}{2}\sin 3x + \frac{1\cdot 3}{2\cdot 4}\sin 5x - \&c =$ [illegible]

$\frac{1}{2}\sin 2x - \frac{1\cdot 3}{2\cdot 4}\sin 4x + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 6x - \&c =$ [illegible]

ii. $\cos x - \frac{1}{2}\cos 3x + \frac{1\cdot 3}{2\cdot 4}\cos 5x - \&c$ [illegible]

$= 1 - \frac{1}{2}\cos 2x + \frac{1\cdot 3}{2\cdot 4}\cos 4x - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 6x + \&c =$ [illegible]

iii. $\sin 2x - \frac{1}{2}\sin 4x + \frac{1\cdot 3}{2\cdot 4}\sin 6x - \&c =$ [illegible]

iv. $\cos 2x - \frac{1}{2}\cos 4x + \frac{1\cdot 3}{2\cdot 4}\cos 6x - \&c =$ [illegible]

v. [illegible]

$$\frac{x}{n+}\,\frac{x}{n+1+}\,\frac{x}{n+2+}\cdots\cdots+\frac{x}{n+n}.$$

$$=\left\{1+\frac{n\,x}{\lfloor 1\; n(n+n)}+\frac{(n-1)(n-2)\,x^2}{\lfloor 2\; n(n+1)(n+n)(n+n-1)}+\frac{(n-2)(n-3)(n-4)\,x^3}{\lfloor 3\; n(n+1)(n+2)(n+n)(n+n-1)(n+n-2)}+\&c\right\}$$

$$\div\left\{1+\frac{(n-1)\,x}{\lfloor 1\,(n+1)(n+n)}+\frac{(n-2)(n-3)\,x^2}{\lfloor 2\,(n+1)(n+2)(n+n)(n+n-1)}+\&c\right\}$$

no. of terms being limited.

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\cdots\cdots+\frac{(n-1)x}{1+}\,\frac{nx}{1}.$$

$$\div\; D=1+\frac{n^2}{\lfloor 1}x+\frac{n^2(n-1)^2}{\lfloor 2}x^2+\frac{n^2(n-1)^2(n-2)^2}{\lfloor 3}x^3+\&c$$

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\cdots\cdots+\frac{(n-1)x}{1+}\,\frac{(n-1)x}{1}.$$

$$D=1+\frac{n^2}{\lfloor 1}\left(1-\frac{1}{n}\right)x+\frac{n^2(n-1)^2}{\lfloor 2}\left(1-\frac{2}{n}\right)x^2+\frac{n^2(n-1)^2(n-2)^2}{\lfloor 3}\left(1-\frac{3}{n}\right)x^3+\&c$$

If $\phi(a,n)=1+ax\cdot\frac{1-x^n}{1-x}+a^2x^4\cdot\frac{(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)}+a^3x^9\cdot\frac{(1-x^{n-2})(1-x^{n-3})(1-x^{n-4})}{(1-x)(1-x^2)(1-x^3)}+\&c$, then

$$\frac{\phi(ax,\,n-1)}{\phi(a,\,n)}=\frac{1}{1+}\,\frac{ax}{1+}\,\frac{ax^2}{1+}\,\frac{ax^3}{1+}\cdots\cdots+\frac{ax^n}{1}.$$

If $\int_0^{\alpha i}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}=i\int_0^{\beta}\frac{d\phi}{\sqrt{1-(1-x)\sin^2\phi}}$

then $\alpha=\log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)$.

vi. $\frac{\cos 2x}{2} - \frac{1}{2}\cdot\frac{\cos 4x}{4} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6} - \&c = \cos\frac{x}{2}\sqrt{2\cos x} - 1$

vii. $\frac{\sin x}{1} - \frac{1}{2}\cdot\frac{\sin 3x}{3} + \frac{1.3}{2.4}\cdot\frac{\sin 5x}{5} - \&c = \sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right)$.

viii. $\frac{\cos x}{1} - \frac{1}{2}\cdot\frac{\cos 3x}{3} + \frac{1.3}{2.4}\cdot\frac{\cos 5x}{5} - \&c = \log_e\left(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2}\right)$

ix. $\frac{\sin 2x}{2^2} - \frac{1}{2}\cdot\frac{\sin 4x}{4^2} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^2} - \&c = \sin\frac{x}{2}\sqrt{2\cos x} + \sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right) - x$.

x. $\frac{\cos 2x}{2^2} - \frac{1}{2}\cdot\frac{\cos 4x}{4^2} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^2} - \&c = \cos\frac{x}{2}\sqrt{2\cos x} - \log_e\left(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2}\right) - 1 + \log_e 2$.

Similarly we can form similar identities for the II part also. From the identities of the I part the following theorem is obtained.

7. Let $F(n) = \left\{\frac{\sin x}{1^n} - \frac{1}{2}\cdot\frac{\sin 3x}{3^n} + \frac{1.3}{2.4}\cdot\frac{\sin 5x}{5^n} - \&c\right\}$
$- \cos\pi n\left\{\left(\frac{\sin 2x}{2^n} - \frac{1}{2}\cdot\frac{\sin 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^n} - \&c\right) - \left(\frac{\sin 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\sin 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^{n+1}} - \&c\right)\right\}$ and

$\psi(n) = \left\{\frac{\cos x}{1^n} - \frac{1}{2}\cdot\frac{\cos 3x}{3^n} + \frac{1.3}{2.4}\cdot\frac{\cos 5x}{5^n} - \&c\right\}$
$+ \cos\pi n\left\{\left(\frac{\cos 2x}{2^n} - \frac{1}{2}\cdot\frac{\cos 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^n} - \&c\right) - \left(\frac{\cos 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\cos 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^{n+1}} - \&c\right)\right\}$, then

If n is odd

1. $\frac{F(n)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor n}S_0\phi(1) - \frac{x^{n-2}}{\lfloor n-2}\left\{S_0\phi(2) + \frac{S_2}{2^2}\phi(1)\right\}$
$+ \frac{x^{n-4}}{\lfloor n-4}\left\{S_0\phi(3) + \frac{S_2}{2^2}\phi(2) + \frac{S_4}{2^4}\phi(1)\right\} - \&c$

$= \frac{A_{n-1}}{\lfloor n-1}\phi(1) - \frac{A_{n-3}}{\lfloor n-3}\phi(2) + \frac{A_{n-5}}{\lfloor n-5}\phi(3) - \&c$

2. $\frac{F(n+1)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor n}S_0\phi(1) - \frac{x^{n-2}}{\lfloor n-2}\left\{S_0\phi(2) + \frac{S_2}{2^2}\phi(1)\right\}$

$$\frac{(1-e^{-x})}{1^2} + \frac{(1-e^{-x})^2}{2^2} + \frac{(1-e^{-x})^3}{3^2} + \&c$$

$$= x - \frac{x^2}{4} + B_2 \frac{x^3}{\lfloor 3} - B_4 \frac{x^5}{\lfloor 5} + B_6 \frac{x^7}{\lfloor 7} - \&c$$

$$\begin{cases} S = \dfrac{\sin\frac{\theta}{2}}{\sinh\frac{y}{2}} + \dfrac{\sin\frac{3\theta}{2}}{\sinh\frac{3y}{2}} + \dfrac{\sin\frac{5\theta}{2}}{\sinh\frac{5y}{2}} + \&c \\ C = \dfrac{\cos\frac{\theta}{2}}{\cosh\frac{y}{2}} + \dfrac{\cos\frac{3\theta}{2}}{\cosh\frac{3y}{2}} + \dfrac{\cos\frac{5\theta}{2}}{\cosh\frac{5y}{2}} + \&c \\ C_1 = \dfrac{1}{2} + \dfrac{\cos\theta}{\cosh y} + \dfrac{\cos 2\theta}{\cosh 2y} + \dfrac{\cos 3\theta}{\cosh 3y} + \&c \end{cases}$$

$$CS = \frac{\sin\theta}{\cosh y} + \frac{2\sin 2\theta}{\cosh 2y} + \frac{3\sin 3\theta}{\cosh 3y} + \&c$$

$$\therefore CS + \frac{dC_1}{d\theta} = 0 \quad \therefore C_1 S + \frac{dC}{d\theta} = 0 \;\&\; CC_1 = \frac{dS}{d\theta}.$$

$$\therefore C^2 + S^2 = x\frac{z^2}{4} \;\&\; C_1^2 + S^2 = \frac{z^2}{4}.$$

$$\therefore \text{Let } C = \sqrt{x}\cdot\frac{z}{2}\cos\phi \;\&\; S = \sqrt{x}\cdot\frac{z}{2}\sin\phi.$$

$$\therefore C_1 = \frac{z}{2}\sqrt{1 - x\sin^2\phi}$$

$$\therefore \frac{z}{2}\cos\phi\sqrt{1-x\sin^2\phi} = \frac{d\sin\phi}{d\theta} = \cos\phi\frac{d\phi}{d\theta}$$

$$\therefore \theta = \frac{2}{z}\int_0^{\phi}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}.$$

$$\left\{\begin{array}{l} \sin\phi \text{ to } i\tan\phi \\ \cos\phi \text{ to } \sec\phi \\ \phi \text{ to } i\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right) \end{array}\right. \qquad \left.\begin{array}{l} \theta \text{ to } i\theta\frac{z}{z'} \\ y \text{ to } y' \\ x \text{ to } 1-x \\ z \text{ to } z' \end{array}\right\}$$

$+ \frac{[illegible]}{[illegible]} \left\{ S_0 \phi(5) + \frac{S_2}{2!} \phi(3) + \frac{S_4}{4!} \phi(1) \right\}$ &c

$= \frac{A_n}{\lfloor n-1} \phi(1) - \frac{A_{n-2}}{\lfloor n-3} \phi(3) + \frac{A_{n-4}}{\lfloor n-5} \phi(5)$ &c II

where $S_n = \frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} +$ &c

$\frac{\pi}{2} \phi(x) = \frac{1}{1^{n+1}} + \frac{x}{[illegible]} \frac{1}{3^{n+1}} + \frac{[illegible]}{[illegible]} \frac{1}{5^{n+1}} +$ &c

and $\frac{2}{\pi} A_n = \left(\frac{\pi}{2}\right)^n + \left(\frac{3\pi}{2}\right)^n + \left(\frac{5\pi}{2}\right)^n + \ldots \left(x \frac{\pi}{2}\right)^n$

If n is even $\frac{\psi(n+1)}{2} \cos \frac{\pi n}{2} = I$ & $\frac{\psi(n+1)}{2} \cos \frac{\pi n}{2} = II$

8. If $\phi(x) = \frac{x}{1} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} +$ &c, then

i. $\phi(x-1) + \phi\left(\frac{1}{x} - 1\right) = \frac{1}{2}(\log_e x)^2$

ii. $\phi(x) + \phi\left(\frac{1}{x}\right) = \frac{1}{2}(\log_e x)^2 + \frac{\pi^2}{6}$

iii. $\phi(-x) + \phi(x-1) = \log_e x \log_e(1-x) - \frac{\pi^2}{6}$

iv. $\phi(x) + \phi(-x) = \frac{1}{2}\phi(-x^2)$

v. $\phi(x) - \phi(-x) = \psi(x) = 2\left(\frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} +\text{&c}\right)$

$\psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \frac{\pi^2}{4} + \log_e x \log_e \frac{1+x}{1-x}$

Ex 1. $\frac{1}{1^2}\cdot\frac{1}{2} + \frac{1}{2^2}\cdot\frac{1}{2^2} + \frac{1}{3^2}\cdot\frac{1}{2^3} + \frac{1}{4^2}\cdot\frac{1}{2^4} +$ &c $= \frac{\pi^2}{12} - \frac{1}{2}(\log_e 2)^2$

2. $\frac{1}{1^2}\left(\frac{\sqrt{5}-1}{2}\right) + \frac{1}{2^2}\left(\frac{\sqrt{5}-1}{2}\right)^2 + \frac{1}{3^2}\left(\frac{\sqrt{5}-1}{2}\right)^3 +$ &c $= \frac{\pi^2}{10} - \left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

3. $\frac{1}{1^2}\left(\frac{3-\sqrt{5}}{2}\right) + \frac{1}{2^2}\left(\frac{3-\sqrt{5}}{2}\right)^2 + \frac{1}{3^2}\left(\frac{3-\sqrt{5}}{2}\right)^3 +$ &c $= \frac{\pi^2}{15} - \left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

4. $\frac{\sqrt{2}-1}{1^2} + \frac{(\sqrt{2}-1)^3}{3^2} + \frac{(\sqrt{2}-1)^5}{5^2} +$ &c $= \frac{\pi^2}{16} - \frac{1}{4}\{\log_e(1+\sqrt{2})\}^2$

5. $\frac{1}{1^2}\left(\frac{\sqrt{5}-1}{2}\right) + \frac{1}{3^2}\left(\frac{\sqrt{5}-1}{2}\right)^3 + \frac{1}{5^2}\left(\frac{\sqrt{5}-1}{2}\right)^5 +$ &c $= \frac{\pi^2}{12} - \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

6. $\frac{\sqrt{5}-2}{1^2} + \frac{(\sqrt{5}-2)^3}{3^2} + \frac{(\sqrt{5}-2)^5}{5^2} +$ &c $= \frac{\pi^2}{24} - \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

[illegible]

If $F(\alpha, \beta) = \frac{\alpha}{n+} \frac{(\beta)^2}{n+} \frac{(2\alpha)^2}{n+} \frac{(3\beta)^2}{n+\&c}$

~~then $F\alpha, \beta$~~ & A, G the ~~A.M. &~~ G.M. between α & β

then $F(A, G)$ is the A.M between $F(\alpha, \beta)$ & $F(\beta, \alpha)$

If $a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = \&c = p$

$a_2 a_3 = b_2 b_3 = c_2 c_3 = d_2 d_3 \&c = q$

$a_3 + a_4 = b_3 + b_4 = c_3 + c_4 = \&c = r$

$a_4 a_5 = b_4 b_5 = c_4 c_5 = \&c = s$ &c by

then $\Sigma \frac{a}{1+} \frac{a_1}{1+} \frac{a_2}{1+} \frac{a_3}{1+\&c}$

$$= \Sigma a - \frac{\Sigma a a_1}{1+p-} \frac{q}{1+r-} \frac{s}{1+t-} \frac{u}{1+v-\&c}$$

If $F(\alpha, \beta) = \tan^{-1} \frac{\alpha}{x+} \frac{\beta^2 + k^2}{x+} \frac{\alpha^2+(2k)^2}{x+} \frac{\beta^2+(3k)^2}{x+\&c}$

and A the A.M between α & β, then.

$F(A, A)$ is the A.M between $F(\alpha, \beta)$ & $F(\beta, \alpha)$.

i. $\phi(x-1) + \phi(\frac{1}{x}-1) + \phi(-x) = \frac{1}{2}(\log_e x)^2 \log_e(1-x) - \frac{1}{6}(\log_e x)^3 - \frac{\pi^2}{6}\log_e x$
$- S_3$.

ii. $\phi(x) - \phi(\frac{1}{x}) = \frac{1}{6}(\log_e x)^3 + \frac{\pi^2}{6}\log_e x$

iii. $\phi(x) + \phi(-x) = \frac{1}{4}\phi(-x^2)$.

Ex 1. $\frac{1}{1^3}\cdot\frac{1}{2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{3^3}\cdot\frac{1}{2^3} + \&c = \frac{1}{6}(\log_e 2)^3 - \frac{\pi^2}{12}\log_e 2$
$+ (\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c)$

2. $\frac{1}{1^3}\cdot\frac{3-\sqrt{5}}{2} + \frac{1}{2^3}\cdot(\frac{3-\sqrt{5}}{2})^2 + \frac{1}{3^3}(\frac{3-\sqrt{5}}{2})^3 + \&c$
$= \frac{2}{3}(\log_e \frac{\sqrt{5}+1}{2})^3 - \frac{2\pi^2}{15}\log_e\frac{\sqrt{5}+1}{2} + S_3$

10. If $\phi(x) = x + (1+\frac{1}{2})\frac{x^2}{2} + (1+\frac{1}{2}+\frac{1}{3})\frac{x^3}{3} + \&c$
then $\phi(\frac{x}{2-x}) = \frac{1}{8}\{\log_e(1-x)\}^2 + \frac{1}{2}(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c)$

Ex 1. $\frac{1}{3} + \frac{1+\frac{1}{2}}{2}\cdot\frac{1}{3^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3}\cdot\frac{1}{3^3} + \&c = \frac{\pi^2}{24} - \frac{1}{8}(\log_e 2)^2$

2. $\frac{1}{5} + \frac{1+\frac{1}{2}}{2}\cdot\frac{1}{5^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3}\cdot\frac{1}{5^3} + \&c = \frac{\pi^2}{20\sqrt{5}}$
$- \frac{3}{\sqrt{5}}(\log_e [illegible])^2$

3. $(\sqrt{5}-2) + \frac{1+\frac{1}{2}}{2}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{2}+\frac{1}{3}}{3}(\sqrt{5}-2)^5 + \&c$
$= \frac{\pi^2}{30} - \frac{3}{4}(\log_e\frac{\sqrt{5}+1}{2})^2$

11. If $\phi(x) = \frac{x^2}{2^2} + (1+\frac{1}{2})\frac{x^3}{3^2} + (1+\frac{1}{2}+\frac{1}{3})\frac{x^4}{4^2} + \&c$ then

i. $\phi(1-x) = \frac{1}{2}\log_e(1-x)(\log_e x)^2 + \log_e x(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c)$
$- (\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c) + S_3$

ii. $\phi(1-x) - \phi(1-\frac{1}{x}) = \frac{1}{6}(\log_e x)^3$

iii. $\phi(1-x) = \frac{1}{2}\log_e(1-x)(\log_e x)^2 - \frac{1}{3}(\log_e x)^3 - \log_e x(\frac{1}{1^2 x} + \frac{1}{2^2 x^2} + \&c)$
$- (\frac{1}{1^3 x} + \frac{1}{2^3 x^2} + \frac{1}{3^3 x^3} + \&c) + S_3$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = 1+\frac{n}{1+}\,\frac{1(n-1)}{3+}\,\frac{2(n-2)}{5+}\,\frac{3(n-3)}{7+\&c}$$

$$= 2+\frac{n-1}{2+}\,\frac{1(n-2)}{4+}\,\frac{2(n-3)}{6+}\,\frac{3(n-4)}{8+\&c}$$

$$= \frac{e^n\,\underline{|n}}{n^n} - \frac{2n}{2+}\,\frac{3n}{3+}\,\frac{4n}{4+}\,\frac{5n}{5+}$$

$$\frac{\operatorname{cosech}\frac{y}{2}}{1+x^2} - \frac{\operatorname{cosech}\frac{3y}{2}}{1+(3x)^2} + \frac{\operatorname{cosech}\frac{5y}{2}}{1+(5x)^2} - \&c$$

$$= \frac{\frac{z}{2}\sqrt{x}}{1+}\,\frac{(1-x)(nz)^2}{1-}\,\frac{x(2nz)^2}{1+}\,\frac{(1-x)(3nz)^2}{1-\&c}.$$

$$\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\cdot\frac{\cos\phi}{1-x\sin^2\phi}\,d\phi$$

$$= \frac{1}{n+}\,\frac{1-x}{n-}\,\frac{4x}{n+}\,\frac{9(1-x)}{n-}\,\frac{16x}{n+\&c}$$

If $1+\sqrt{1-\alpha} = \frac{2}{1+\sqrt{\beta}} = \gamma$

then $\frac{\sqrt{\beta}}{n+}\,\frac{1^2}{n+}\,\frac{2^2\beta}{n+}\,\frac{3^2}{n+}\,\frac{4^2\beta}{n+}\,\frac{5^2}{n+}\ \&c$

$+\frac{1}{n+}\,\frac{1^2\beta}{n+}\,\frac{2^2}{n+}\,\frac{3^2\beta}{n+}\,\frac{4^2}{n+}\ \&c$

$= \frac{2}{n\gamma+}\,\frac{1^2\alpha}{n\gamma+}\,\frac{2^2}{n\gamma+}\,\frac{3^2\alpha}{n\gamma+}\,\frac{4^2}{n\gamma+}\ \&c$

iv. $\phi(x) + \phi\left(-\frac{1}{x}\right) = -\frac{1}{6}(\log_e x)^3 + \log_e x \left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right)$

$- \left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right) + S_3$

12. If $\phi(x) = \frac{x^2}{2^3} - \left(1+\frac{1}{2}\right)\frac{x^3}{3^3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4^3} - \&c$, then

i. $\phi(x-1) - \phi\left(\frac{1}{x}-1\right) = -\frac{1}{24}(\log_e x)^4 + \frac{1}{6}(\log_e x)^3 \log_e(1-x)$

$+ \log_e x \left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) - 2\left(\frac{x}{1^4} + \frac{x^2}{2^4} + \frac{x^3}{3^4} + \&c\right)$

$+ S_3 \log_e x + \frac{\pi^4}{45}$

ii. $\phi(x) - \phi\left(\frac{1}{x}\right) = -\frac{1}{24}(\log_e x)^4 + \log_e x \left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right)$

$- 2\left(\frac{x}{1^4} - \frac{x^2}{2^4} + \frac{x^3}{3^4} - \&c\right) + S_3 \log_e x + \frac{7\pi^4}{360}$.

13. If $\phi(x) = \frac{x^2}{2^2} + \left(1+\frac{1}{3}\right)\frac{x^4}{4^2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6^2} + \&c$ and

$\psi(x) = \frac{x^2}{2^3} + \left(1+\frac{1}{3}\right)\frac{x^4}{4^3} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6^3} + \&c$ then

i. $\phi\left(\frac{1-x}{1+x}\right) = \frac{1}{2}(\log_e x)^2 \log_e \frac{1-x}{1+x} + \frac{1}{2}\log_e x \left(\frac{x}{1^2} + \frac{x^3}{3^2} + \&c\right)$

$+ \frac{1}{2}\left(\frac{1-x}{1^3} + \frac{1-x^3}{3^3} + \frac{1-x^5}{5^3} + \&c\right)$

ii. $\psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \phi(x)\log_e x + \phi\left(\frac{1-x}{1+x}\right)\log_e \frac{1-x}{1+x}$

$-\frac{1}{16}(\log_e x)^2\left(\log_e \frac{1-x}{1+x}\right)^2$

$+ \frac{\pi^2}{8}\left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right) - [illegible]\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

14. If $\phi(x) = x + \left(1+\frac{1}{2}\right)\frac{x^3}{3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^5}{5} + \&c$ then

$\phi\left(\frac{1-x}{1+x}\right) = [illegible](1-\log_e x)\log_e x + \frac{1+x}{1-x}\log_e \frac{2x}{(1+x)^2} + \frac{1}{4}(\log_e x)^2$

$- \left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right) + \frac{\pi^2}{12}$

Ex. 1. $\frac{1}{2} + \frac{1}{2^2}\ \frac{1+\frac{1}{2}}{2^2}\ [illegible] + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2}\ [illegible] + \&c = S_3 - [illegible]$

$\frac{1}{1^3} + \frac{1+\frac{1}{2}+\frac{1}{3}}{[illegible]} + [illegible] + \&c = [illegible]$

If $I(n)$ be the nearest integer to

$$\frac{1}{4n}\left\{\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right\}$$

then $I(0) + xI(1) + x^2 I(2) + x^3 I(3) + \&c$

$$= \frac{1}{1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \&c}$$

If $\int_0^\infty \frac{e^{-m^2x^2}}{1+x^2}\,dx = \phi(m)$, & if $m \nless n$.

then $\int_0^\infty \frac{e^{-m^2x^2}}{1+x^2}\cos 2mnx\,dx = \frac{1}{2}e^{-n^2}\left\{\phi(m+n) + \phi(m-n)\right\}$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = \frac{e^n \lfloor n}{2n^n} + \frac{2}{3} - \frac{4}{135n} + \frac{8}{27.105n^2} + \frac{16}{105.81n^3} \ast \ \text{~~\&c~~}$$

$$(m-n-1)\int_0^\infty \frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{x}{m}\right)^m}dx = \frac{m}{2}\cdot\frac{m^m \lfloor n}{n^n \lfloor m}\cdot\frac{\lfloor m-n}{(m-n)^{m-n}}$$

$$+ \frac{2}{3}(m+n) - \frac{4(m+n)(m-2n)(m-\frac{n}{2})}{135mn(m-n)}$$

$$+ \frac{8(m^3+n^3)(m-2n)(m-\frac{n}{2})}{27.105m^2n^2(m-n)^2} + \frac{16(m^3+n^3)}{105.81m^2n^2} \times \frac{(m^2-mn+n^2)(m-2n)(m-\frac{n}{2})}{(m-n)^3}$$

$$- \frac{32.281}{3^8.5^2.7.11n^4} - \&c$$

3. $\frac{1}{1^2} + \frac{1+\frac{1}{2}}{2^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2} + \&c = 2\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \&c\right)$ [illegible]

$(\sqrt{5}-2) + \frac{1+\frac{1}{2}}{3}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{2}+\frac{1}{3}}{5}(\sqrt{5}-2)^5 + \&c$

$= \frac{\pi^2}{12} + \frac{3}{2}\left(\log\frac{\sqrt{5}+1}{2}\right)^2 + (\sqrt{5}+2)\log_e 4 + (3\sqrt{5}+5+\log 2)\log_e\frac{\sqrt{5}-1}{2}$ [illegible]

15. $S_{n+1}\cos\frac{\pi n}{2}\,\lfloor n = \int \frac{x^n}{2}\cot\frac{x}{2}\,dx +$

$x^n\left(\frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \&c\right)$

$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \&c\right)$

$- n(n-1)x^{n-2}\left(\frac{\cos x}{1^3} + \frac{\cos 2x}{2^3} + \frac{\cos 3x}{3^3} + \&c\right)$

$+ n(n-1)(n-2)x^{n-3}\left(\frac{\sin x}{1^4} + \frac{\sin 2x}{2^4} + \frac{\sin 3x}{3^4} + \&c\right)$

$+ \&c$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ & $S_1 = \log 2$.

Sol. $\sin x + \sin 2x + \sin 3x + \&c = \frac{1}{2}\cot\frac{x}{2}$

$\int x^n(\sin x + \sin 2x + \&c)\,dx = \int \frac{x^n}{2}\cot\frac{x}{2}\,dx$

and apply $\int uv\,dx = uv_1 - u'v_2 + u''v_3$ &c.

16. $S_{n+1}\cos\frac{\pi n}{2}\,\lfloor n = \int \frac{x^n}{2\sin x}\,dx$

$+ x^n\left(\frac{\cos x}{1} + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \&c\right)$

$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \&c\right)$

$- n(n-1)x^{n-2}\left(\frac{\cos x}{1^3} + \frac{\cos 3x}{3^3} + \frac{\cos 5x}{5^3} + \&c\right) + \&c$

Sol. $\sin x + \sin 3x + \sin 5x + \&c = \frac{1}{2}\operatorname{cosec} x$.

17. $\int x^n \cot x\,dx = f(x)$ then

$\frac{\pi}{4}(\ldots) = \pi^n\{f(2\pi) - f(\pi)\}$ [illegible]

$+ \frac{n(n-1)}{\lfloor 2}\pi^{n-2}\{f(2x) - 2^{n}f(x)\}$ [illegible]

$$\frac{\operatorname{Sech}\frac{y}{2}}{1+n^2}+\frac{\operatorname{Sech}\frac{3y}{2}}{1+(3n)^2}+\frac{\operatorname{Sech}\frac{5y}{2}}{1+(5n)^2}+\&c$$

$$=\frac{1}{2}\cdot\cfrac{z\sqrt{x}}{1+\cfrac{(nz)^2}{1+\cfrac{(2nz)^2x}{1+\cfrac{(3nz)^2}{1+\cfrac{(4nz)^2x}{1+\&c}}}}}$$

$$\int_0^\infty e^{-n\int_0^\phi\frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\cdot\frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}\,d\phi=\cfrac{1}{n+\cfrac{1}{n+\cfrac{4x}{n+\cfrac{9}{n+\cfrac{16x}{n+\&c}}}}}$$

$$\frac{\operatorname{Sech}\frac{y}{2}}{1+n^2}-\frac{3\operatorname{Sech}\frac{3y}{2}}{1+(3n)^2}+\frac{5\operatorname{Sech}\frac{5y}{2}}{1+(5n)^2}-\&c$$

$$=\frac{1}{2}\cdot\cfrac{z^2\sqrt{x(1-x)}}{1+n^2z^2(1-2x)+\cfrac{2^2(2^2-1)x(1-x)n^4z^4}{1+(3nz)^2(1-2x)+\cfrac{4^2(4^2-1)x(1-x)n^4z^4}{1+(5nz)^2(1-2x)+\&c}}}$$

$$\frac{\operatorname{Cosech}\frac{y}{2}}{1+n^2}+\frac{3\operatorname{Cosech}\frac{3y}{2}}{1+(3n)^2}+\frac{5\operatorname{Cosech}\frac{5y}{2}}{1+(5n)^2}+\&c$$

$$=\frac{1}{2}\cdot\cfrac{z^2\sqrt{x}}{1+(nz)^2(1+x)-\cfrac{2^2(2^2-1)xn^4z^4}{1+(3nz)^2(1+x)-\cfrac{4^2(4^2-1)xn^4z^4}{1+(5nz)^2(1+x)-\&c}}}$$

Sol. $\tan x = \cot x - 2\cot 2x$ and

$\phi_n(\frac{\pi}{2}-x) = \int(\frac{\pi}{2}-x)^n \cot(\frac{\pi}{2}-x)\, d(\frac{\pi}{2}-x) = -\int(\frac{\pi}{2}-x)^n \tan x\, dx$

N.B. Let $\sin x = y$ & $\tan x = z$, then

$\int x^n \cot x\, dx = \int \frac{x^n}{\sin x}\cos x\, dx = \int \frac{(\sin^{-1}y)^n}{y}\, dy$ and

$\int \frac{2x^n}{\sin 2x}\, dx = \int \frac{x^n}{\cos x \sin x}\, dx = \int \frac{x^n}{\tan x}\sec^2 x\, dx$

$= \int \frac{(\tan^{-1}z)^n}{z}\, dz$. So we have to find $(\sin^{-1}y)^n$ & $(\tan^{-1}y)^n$

i. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c$

ii. $\frac{1}{2}(\tan^{-1}x)^2 = \frac{x^2}{2} - (1+\frac{1}{3})\frac{x^4}{4} + (1+\frac{1}{3}+\frac{1}{5})\frac{x^6}{6} - \&c$

iii. $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5} + \&c$

iv. $\frac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2\cdot4}{3\cdot5}\cdot\frac{x^6}{6} + \frac{2\cdot4\cdot6}{3\cdot5\cdot7}\cdot\frac{x^8}{8} + \&c$

v. $\frac{1}{6}(\sin^{-1}x)^3 = \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5}(1+\frac{1}{3^2}) + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}(1+\frac{1}{3^2}+\frac{1}{5^2})$
$+ \&c$

vi. $\frac{1}{24}(\sin^{-1}x)^4 = \frac{2}{3}\cdot\frac{x^4}{4}\cdot\frac{1}{2^2} + \frac{2\cdot4}{3\cdot5}\cdot\frac{x^6}{6}(\frac{1}{2^2}+\frac{1}{4^2}) + \&c$

vii. $\frac{d(\sin^{-1}x)^n}{dx} = n\frac{(\sin^{-1}x)^{n-1}}{\sqrt{1-x^2}}$ & $\frac{d(\tan^{-1}x)^n}{dx} = n\frac{(\tan^{-1}x)^{n-1}}{1+x^2}$

18. $\frac{\sin x}{1^2} + \frac{1}{2}\cdot\frac{\sin^3 x}{3^2} + \frac{1\cdot3}{2\cdot4}\cdot\frac{\sin^5 x}{5^2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{\sin^7 x}{7^2} + \&c$

$= x\log_e 2\sin x + \frac{1}{2}(\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \&c)$

Ex 1. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7^2} + \&c = \frac{\pi}{2}\log_e 2$

2. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{1}{2} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^2}\cdot\frac{1}{2^2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7^2}\cdot\frac{1}{2^3} + \&c$

$= \frac{\pi}{4\sqrt{2}}\log_e 2 + \frac{1}{\sqrt{2}}(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c)$

3. $\frac{1}{1^2} + \frac{1}{2}$ [illegible] $+ \frac{1\cdot3}{2\cdot4}$ [illegible] $+ \&c$

$= \frac{3\sqrt{3}}{4}(\frac{1}{1^2}$ [illegible] $+ \&c) + \frac{\pi}{3\sqrt{3}}$ [illegible]

If $\alpha\beta = 4\pi^2$ & $F(x) = \frac{\sqrt{5}+1}{2} + \frac{\sqrt[5]{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+\&c}$

then $F(e^{-\alpha})\,F(e^{-\beta}) = \frac{5+\sqrt{5}}{2}$.

$$f^2(-x^2, -x^3) - \sqrt[5]{x^2}\, f^2(-x, -x^4)$$
$$= f(-x, -x^2)\left\{f(-\sqrt[5]{x}, -\sqrt[5]{x^2}) + \sqrt[5]{x}\, f(-x^5, -x^{10})\right\}.$$

If $K = \frac{f(-x^{\frac{1}{5}}, -x^{\frac{2}{5}})}{\sqrt[5]{x}\, f(-x^5, -x^{10})}$, then

$$\frac{\sqrt{K^2+2K+5} - (K+1)}{2} = \frac{\sqrt[5]{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+\&c}.$$

If $\int_0^\infty \phi(x) \cos 2nx \, dx = \frac{\sqrt{\pi}}{2}\psi(n)$

then $\int_0^\infty e^{-x^2}\phi(x)\,dx = \int_0^\infty e^{-x^2}\psi(x)\,dx$

&c &c &c

$$\int_0^\infty \frac{F(a+bx) - F(a-bx)}{\sinh}$$

4. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2}\left(\frac{3}{4}\right) + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5^2}\left(\frac{3}{4}\right)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7^2}\left(\frac{3}{4}\right)^3 + \&c$

$= \frac{\pi}{3\sqrt{3}}\log_e 3 - \frac{2\pi^2}{27} + \left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right)$

19. $\frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \frac{\tan^7 x}{7^2} + \&c$

$= \log_e \tan x + \frac{1}{2}\left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \&c\right)$

Ex. 1. $\frac{1}{1^2}\cdot\frac{1}{3} - \frac{1}{3^2}\cdot\frac{1}{3^2} + \frac{1}{5^2}\cdot\frac{1}{3^3} - \&c$

$= -\frac{\pi}{12\sqrt{3}}\log_e 3 - \frac{5}{64}\pi^2 + \frac{5}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right)$

2. $\frac{\sqrt{2}-1}{1^2} - \frac{(\sqrt{2}-1)^3}{3^2} + \frac{(\sqrt{2}-1)^5}{5^2} - \frac{(\sqrt{2}-1)^7}{7^2} + \&c$

$= -\frac{\pi}{8}\log_e(1+\sqrt{2}) - \frac{\pi^2}{16} + \sqrt{2}\left(\frac{1}{1^2} - \frac{1}{5^2} + \frac{1}{9^2} - \frac{1}{13^2} + \&c\right)$

3. $\frac{2-\sqrt{3}}{1^2} - \frac{(2-\sqrt{3})^3}{3^2} + \frac{(2-\sqrt{3})^5}{5^2} - \&c = -\frac{\pi}{12}\log_e(2+\sqrt{3})$

$+ \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right)$

20. If x lies between 0 & $\frac{\pi}{4}$

$\frac{\cos x - \sin x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x - \sin^3 x}{3^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos^5 x - \sin^5 x}{5^2} + \&c$

$+ \frac{1}{2}\left\{\frac{\sin 2x}{1^2} + \frac{1}{2}\cdot\frac{\sin^3 2x}{3^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\sin^5 2x}{5^2} + \&c\right\}$

$= \frac{\pi}{2}\log_e 2\cos x$.

Ex. If $\psi(x) = \frac{x}{1^2} + \frac{1}{2}\cdot\frac{x^3}{3^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5^2} + \&c$

shew that $\psi\left(\frac{3}{5}\right) - \frac{1}{2}\psi\left(\frac{24}{25}\right) = \frac{\pi}{2}\log_e 2$ [illegible]

21. $\frac{\cos x + \sin x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x + \sin^3 x}{3^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos^5 x + \sin^5 x}{5^2} + \&c$ [illegible]

$= \frac{\pi}{2}\log_e 2\cos x + \frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \&c$

Ex. $\frac{1}{1^2}$ [illegible] $+ \frac{1}{2}\cdot$ [illegible] $+ \frac{1\cdot 3}{2\cdot 4}$ [illegible] $+ \&c$

$= \frac{\pi}{2}$ [illegible] $+ \sqrt{2}\left(\frac{1}{1^2}\right.$ [illegible]

22. $\frac{\sin^2 x}{1^2}$ [illegible]

$$x + \frac{x^2+1}{2x+}\;\frac{x^2+9}{2x+}\;\frac{x^2+25}{2x+\&c}$$

$$= x + \frac{x^2-1}{2x+}\;\frac{x^2-9}{2x+}\;\frac{x^2-25}{2x+\&c} \times \frac{1-e^{-\pi x}}{1-2e^{-\frac{\pi x}{2}}\sin\frac{\pi x}{2}+e^{-\pi x}}$$

$$\frac{1+\dfrac{ax^2}{1-x}+\dfrac{a^2x^6}{(1-x)(1-x^2)}+\dfrac{a^3x^{12}}{(1-x)(1-x^2)(1-x^3)}+\&c}{1+\dfrac{ax}{1-x}+\dfrac{a^2x^4}{(1-x)(1-x^2)}+\dfrac{a^3x^9}{(1-x)(1-x^2)(1-x^3)}+\&c}$$

$$= \frac{1}{1+}\;\frac{ax}{1+}\;\frac{ax^2}{1+}\;\frac{ax^3}{1+}\;\frac{ax^4}{1+\&c}$$

$$\frac{1}{(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)\,\&c}$$

$$= 1 + \frac{ax}{(1-x)(1-ax)} + \frac{a^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)}$$

$$+ \frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)(1-ax)(1-ax^2)(1-ax^3)} + \&c$$

$$\frac{f(-x^5,-x^{10})}{f(-x,-x^4)} = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \&c$$

$$\frac{f(-x^5,-x^{10})}{f(-x^2,-x^3)} = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \&c.$$

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$= \frac{x^2}{2} \log \sin x + \frac{x}{2}\left(\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \&c\right)$ [illegible]

$+ \frac{1}{4}\left(\frac{\cos 2x}{1^3} + \frac{\cos 4x}{2^3} + \frac{\cos 6x}{3^3} + \&c\right)$ [illegible]

Ex 1. $\frac{\cos^2 x}{2^2} + \frac{2}{3} \cdot \frac{\cos^4 x}{4^2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\cos^6 x}{6^2} + \&c$ [illegible]

$= \frac{\pi^2}{8} \log_e 2 - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \&c\right)$ [illegible]

2. [illegible] $\frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{1}{6^2}$ [illegible]

$= \frac{\pi^2}{64} \log 2 + \frac{\pi}{8}$ [illegible]

Ex 3. $\frac{\tan^2 x}{2^2} - \left(1 + \frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$ [illegible]

$= \frac{x^2}{2} \log_e \tan x + \frac{x}{2}\left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \&c\right)$

$+ \frac{1}{4}\left(\frac{\cos 2x}{1^3} + \frac{\cos 6x}{3^3} + \frac{\cos 10x}{5^3} + \&c\right) - \frac{7}{8}\left(\frac{1}{1^3} + \frac{1}{3^3} + \&c\right)$ [illegible]

Ex. $\frac{1}{2^2} - \frac{1 + \frac{1}{3}}{4^2} + \frac{1 + \frac{1}{3} + \frac{1}{5}}{6^2} - \&c$

$= \frac{\pi}{2}\left(\frac{1}{1^2} - \frac{1}{3^2} + \&c\right)$ [illegible]

Ex 4. $\frac{\cos^2 x}{2^2} + \frac{1}{2} \cdot \frac{\sin^2 x}{4^2} + \frac{2}{3} \cdot \frac{\cos^4 x + \sin^4 x}{4^2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\cos^6 x + \sin^6 x}{6^2}$

$+ \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{\cos^8 x + \sin^8 x}{8^2} + \&c$

$= -\frac{\pi^2}{8} \log_e 2\cos x + \frac{\pi}{2}\left\{\frac{\cos x}{1^2} + \frac{1}{2} \cdot \frac{\cos^3 x}{3^2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\cos^5 x}{5^2} + \&c\right\}$

$+ \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3} \cdot \frac{\sin^4 2x}{4^2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\sin^6 2x}{6^2} + \&c\right\}$

$- \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

Ex 5. $\frac{\tan^2 x}{2^2} - \left(1 + \frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$

$= 2\left\{\frac{\sin^2 x}{2^2} + \frac{2}{3} \cdot \frac{\sin^4 x}{4^2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\sin^6 x}{6^2} + \&c\right\}$

$- \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3} \cdot \frac{\sin^4 2x}{4^2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{\sin^6 2x}{6^2} + \&c\right\}$

$$\int_0^\alpha \frac{d\phi}{\sqrt{(1-a\sin^2\phi)(1-b\sin^2\phi)}} = \frac{1}{\sqrt{1-b}}\int_0^\beta \frac{d\phi}{\sqrt{1-\frac{a-b}{1-b}\sin^2\phi}}$$

$$\text{c}\quad \frac{\tan\beta}{\tan\alpha} = \sqrt{1-b}.$$

(i) $\sqrt[24]{y}\,(1-y)(1-y^2)(1-y^3)\,\&c = \dfrac{\sqrt[24]{x}\,\sqrt[8]{1-x}}{\sqrt[8]{3}}\sqrt{1+\frac{1.2}{3^2}x+\&c}$

(ii) $= \dfrac{\sqrt[24]{x}\,\sqrt[24]{1-x}}{\sqrt[6]{2}}\sqrt{1+\frac{1.3}{4^2}x+\&c}$. (iii) $= \sqrt[24]{\dfrac{x(1-x)}{432}}\sqrt{1+\frac{1.5}{6^2}x+\&c}$

$\sqrt[12]{y}\,(1-y^2)(1-y^4)(1-y^6)\,\&c = \dfrac{\sqrt[24]{x}\,\sqrt[24]{1-x}}{\sqrt{2}}\sqrt{1+\frac{1.3}{4^2}x+\&c}$

$\sqrt[8]{y}\,(1-y^3)(1-y^6)(1-y^9)\,\&c = \sqrt[8]{\frac{x}{27}}\,\sqrt[24]{1-x}\,\sqrt{1+\frac{1.2}{3^2}x+\&c}$

$1+240\left(\dfrac{1^3y^3}{1-y^3}+\dfrac{2^3y^6}{1-y^6}+\&c\right) = \left(1+\frac{1.2}{3^2}x+\&c\right)^4\left(1-\frac{8}{9}x\right)$.

$1-504\left(\dfrac{1^5y^3}{1-y^3}+\dfrac{2^5y^6}{1-y^6}+\&c\right) = \left(1+\frac{1.2}{3^2}x+\&c\right)^6\left(1-\frac{4}{3}x+\frac{8}{27}x^2\right)$.

26. $\int_0^{\frac{\pi}{2}} x \cos^n x \sin nx \, dx = \frac{\pi}{2^{n+2}} \left([illegible] \right)$

$\int_0^{\frac{\pi}{2}} \cos^n x \sin nx \, dx = \frac{1}{2^{n+1}} \left(\frac{2}{1} + \frac{2^2}{2} + [illegible] \right)$

These are true for all values of n.

Cor. 1. Show that $\frac{2}{1} + \frac{2^2}{2} + [illegible]$ can be expanded in ascending powers of $\frac{1}{n}$ in a convergent series, the first term being [illegible] then show that

2. If $\phi(x) = \frac{2x}{1} + \frac{2^2 x^2}{2} + [illegible]$

$\phi(-x) + \Sigma \frac{1}{x-1} + [illegible]$

hence show how to find the value of the series

$\frac{1}{n} \frac{1}{2} + [illegible]$

27. $1^r \log 1 + 2^r \log 2 + 3^r \log 3 + \ldots + x^r \log x = \phi_r(x)$

$\phi_r(x) = C_r + (1^r + 2^r + 3^r + \ldots + x^r)\log x - \frac{x^{r+1}}{(r+1)^2} - [illegible]$

$+ \frac{B_2}{\underline{|2}} \cdot r \cdot \frac{1}{r} x^{r-1} - \frac{B_4}{\underline{|4}} r(r-1)(r-2)\left(\frac{1}{r} + \frac{1}{r-1} + \frac{1}{r-2}\right) x^{r-3} + [illegible]$

and $C_{r} = \frac{B_r}{r} \left\{ \cos \frac{\pi r}{2} [illegible] \right\}$

$- \frac{2 \underline{|r}}{(2\pi)^{r}} \cos \frac{\pi r}{2} \left\{ \frac{\log 1}{1^{r+1}} + \frac{\log 2}{2^{r+1}} + \frac{\log 3}{3^{r+1}} + \&c \right\}$

Cor. If r is even $C_r = [illegible]$

$C_0 = \frac{1}{2} \log 2\pi$, $C_1 = [illegible]$

Sol. Divide both sides in 27 by x^r and then differentiate both sides with regard to x.

Ex 1. $\frac{(1^1 \cdot 2^2 \cdot 3^3 \cdots x^x)^2 e^{\frac{1}{6}(x^2 - [illegible])}}{\sqrt[3]{\frac{x^{x(x+3)} \underline{|x^2}}{x + \frac{1}{2}}}}$ when $x = \infty$

$= [illegible]$

$$\phi(1) - \phi(2) + \phi(3) - \&c$$
$$= \phi(0) - \phi(-1) + \phi(-2) - \&c$$

If $\alpha\beta = \frac{\pi}{4}$, then

$$\sqrt{\alpha}\left\{1 - \frac{\alpha^2}{\lfloor 2} + \frac{5}{\lfloor 4}\alpha^4 - \frac{61}{\lfloor 6}\alpha^6 + \&c\right\}$$
$$= \sqrt{\beta}\left\{1 - \frac{\beta^2}{\lfloor 2} + \frac{5}{\lfloor 4}\beta^4 - \frac{61}{\lfloor 6}\beta^6 + \&c\right\}$$

~~$\sqrt{\alpha}\left\{e^{\alpha^2} - e^{9\alpha^2} + e^{25\alpha^2} - \&c\right\}$~~ $= \sqrt{\beta}\left\{e^{\beta^2} - e^{9\beta^2} + e^{25\beta^2} - \&c\right\}$

~~$\int_0^\infty e^{x^2} \sin 2nx\, dx = \frac{\sqrt{\pi}}{2} e^{n^2}.$~~

If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{e^{\alpha x} + e^{-\alpha x}}\, dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{e^{\beta x} + e^{-\beta x}}\, dx$$

$$1 - xa + x^3a^2 - x^6a^3 + x^{10}a^4 - \&c$$
$$= \cfrac{1}{1 + \cfrac{xa}{1 + \cfrac{(x^2 - x)a}{1 + \cfrac{x^3a}{1 + \cfrac{(x^4 - x^2)a}{1 + \cfrac{x^5a}{1 + \&c}}}}}}$$

$$D_{2n} = 1 + a x^n \cdot \frac{1 - x^n}{1 - x} + a^2 x^{2n} \cdot \frac{(1 - x^n)(1 - x^{n-1})}{(1 - x)(1 - x^2)}$$
$$+ a^3 x^{3n} \cdot \frac{(1 - x^n)(1 - x^{n-1})(1 - x^{n-2})}{(1 - x)(1 - x^2)(1 - x^3)} + \&c = \phi(a)$$

$$D_{2n+1} = \phi(ax).$$

28. $\phi_n(x) + n C_{n-1} x + \frac{n(n-1)}{\lfloor 2} C_{n-2} x^2 + \frac{n(n-1)(n-2)}{\lfloor 3} C_{n-3} x^3 + \&c$

$+ C_0 x^n + S_1 \frac{x^{n+1}}{n+1} \ldots - S_2 \frac{x^{n+2} \lfloor 2}{(n+1)(n+2)} + \&c = f(x, n)$

where C_n is the constant of $1^n \log 1 + 2^n \log 2 + \&c$

& S_n is that of $1^n + 2^n + 3^n + \&c$

$f(x, n) = \frac{1^n + 2^n + 3^n \ldots + x^n}{n} + n \int_0^x f(x, n-1)\, dx$

$f(x, n) = (1^n + 2^n + \ldots x^n - S_{-n}) \equiv \frac{x}{n} - \frac{n}{\lfloor 2} B_2 x^{n-1} \ldots$

$\frac{n(n-1)(n-2)}{\lfloor 4} B_4 x^{n-3} (1 + \frac{1}{2} + \frac{1}{3}) - \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 6} B_6$

$x^{n-5} (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}) + \&c$

29. $\phi_n(x) = n^n \{ \phi_n(\frac{x}{n}) + \phi_n(\frac{x-1}{n}) + \ldots + \phi_n(\frac{x-n+1}{n}) \}$

$= (1^n + 2^n + \ldots + x^n) \log n - S_{-n} \log n - (n^{n+1} - 1) C_n$

Cor 1. $\phi_n(-\frac{1}{n}) + \phi_n(-\frac{2}{n}) + \phi_n(-\frac{3}{n}) + \ldots + \phi_n(-\frac{n-1}{n})$

$= \frac{\log n}{n^n} S_{-n} + (n - \frac{1}{n^n}) C_n$

2. $\phi_n(-\frac{1}{2}) = \frac{\log 2}{2^n} S_{-n} + (2 - \frac{1}{2^n}) C_n$

30. If n is even

$\phi_n(x-1) + \phi_n(-x) = 2C_n + \frac{\lfloor n}{(2\pi)^n} \cos\frac{\pi n}{2} \{ \frac{\cos 2\pi x}{1^{n+1}} + \frac{\cos 4\pi x}{2^{n+1}} \ldots$

If n is odd

$\phi_n(x-1) - \phi_n(-x) = \frac{\lfloor n}{(2\pi)^n} \sin\frac{\pi n}{2} \{ \frac{\sin 2\pi x}{1^{n+1}} + \frac{\sin 4\pi x}{2^{n+1}} \ldots \}$

Sol. $\Sigma \frac{1}{x+1} - \Sigma \frac{1}{-x} = -\pi \cot \pi x = -\frac{\pi}{2} (\sin 2\pi x + \ldots$

[illegible]

If $\int_0^h \phi(x)\cos nx\,dx = \psi(n)$ & $\alpha\beta = \frac{\pi}{2}$

then $\alpha\left\{\frac{1}{2}\phi(0) + \phi(\alpha)\cos n\alpha + \phi(2\alpha)\cos 2n\alpha + \ldots\ldots + \phi(m\alpha)\cos mn\alpha\right\}$

$= \psi(n) + \psi(2\beta - n) + \psi(2\beta + n) + \psi(4\beta - n) + \psi(4\beta + n) + \psi(6\beta - n) + \psi(6\beta + n) + \&c \text{ ad. inf.}$

where $m\alpha$ is the greatest multiple of α less than h; but if h be a multiple of α the last term is $\frac{1}{2}\phi(h)\cos nh$; in both the cases n lies between 0 & 2β.

$$\int_0^h \frac{\sin nx}{\sin x}\phi(x)\,dx$$

$$= \pi\left\{\frac{1}{2}\phi(0) - \phi(\pi)\cos n\pi + \phi(2\pi)\cos 2n\pi - \ldots\ldots \pm \phi(m\pi)\cos mn\pi\right\}$$

$$- 2\psi(n+1) - 2\psi(n+3) - 2\psi(n+5) - \&c \text{ ad. inf.}$$

N.B. The general theorems true for all values of x ... got by differentiating 1st 14 & 15 with respect to n.

31. $\frac{1}{1}\log 1 + 2\log 2 + \ldots + x\log x = \phi_1(x)$; then

$\pi\{\phi_1(x-1) - \phi_1(-x)\} = \psi(x) - \pi x \log(2\sin\pi x)$ where

$$\psi(x) = \sin\pi x + \frac{1}{2}\frac{\sin^3\pi x}{3^2} + \frac{1.3}{2.4}\frac{\sin^5\pi x}{5^2} + \&c$$

$$= \tan\pi x - \left(1 + \frac{1}{3}\right)\frac{\tan^3\pi x}{3} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{\tan^5\pi x}{5} - \&c$$

II. $\psi(x) + \psi(\frac{1}{2} - x) = \tan\pi x - \frac{\tan^3\pi x}{3^2} + \frac{\tan^5\pi x}{5^2} - \&c$

$$+ \frac{\pi}{2}\log(2\cos\pi x)$$

III. $\psi(\frac{1}{2} - x) + \frac{1}{2}\psi(2x) - \psi(x) = \frac{\pi}{2}\log(2\cos\pi x)$.

Ex. 1. $\psi(\frac{1}{2}) = \frac{\pi}{2}\log 2$.

2. $\psi(\frac{1}{4}) = \left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right) + \frac{\pi}{4}\log 2$.

3. $\psi(\frac{1}{6}) = \frac{3\sqrt{3}}{4}\left(\frac{1}{1^2} - \frac{1}{5^2} + \frac{1}{7^2} - \&c\right) - \frac{\pi^2}{6\sqrt{3}}$ [illegible]

4. $\psi(\frac{1}{3}) = \frac{\sqrt{3}}{2}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} \ldots \&c\right) - \frac{\pi^2}{9\sqrt{3}} + \ldots \frac{\log 3}{2}$ [illegible]

5. $\psi(\frac{1}{2} - x) + \psi(\frac{1}{2} + x) = \pi \log(2\cos\pi x)$ [illegible]

6. Find $\psi(\frac{1}{8})$, $\psi(\frac{3}{8})$, $\psi(\frac{1}{12})$, $\psi(\frac{5}{12})$ − $\psi(\frac{1}{12})$ [illegible]

7. Find $\psi(\frac{1}{5})$ & $\psi(\frac{2}{5})$

8. $2\psi(x) - \frac{1}{2}\psi(2x) = \tan\pi x - \frac{\tan^3\pi x}{3^2} + \frac{\tan^5\pi x}{5^2} - \&c$

Similarly we can find peculiarities for $\phi_2(x)$, $\phi_3(x)$ &c

32. $\sin 2x + \frac{2}{3^2}\sin^3 2x + \frac{2.4}{3.5^2}\sin^5 2x + \frac{2.4.6}{3.5.7^2}\sin^7 2x + \&c$

$$= 2\left(\frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \&c\right)$$

Cor. $\frac{2x}{1+x^2} + \frac{2}{3^2}\left(\frac{2x}{1+x^2}\right)^3 + \frac{2.4}{3.5^2}\left(\frac{2x}{1+x^2}\right)^5 + \&c$

$$= 2\left(\frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \&c\right)$$

If $\cfrac{a_n}{b_n x + \cfrac{a_{n+1}}{b_{n+1} x + \cfrac{a_{n+2}}{b_{n+2} x + \&c}}} = C_n (1 - P_n x + Q_n x^2 - R_n x^3 + \&c)$

then $C_n C_{n+1} = a_n$; $P_n + P_{n+1} = \frac{b_n}{C_{n+1}}$ or $\frac{b_n C_n}{a_n}$

$Q_n + Q_{n+1} = (P_n)^2$. $R_n + R_{n+1} = P_n (Q_n - Q_{n+1})$.

$S_n + S_{n+1} = P_n (R_n - R_{n+1}) - Q_n Q_{n+1}$.

Generally

$$Z_n + Z_{n+1} = P_n (Y_n - Y_{n+1}) - Q_n X_{n+1} - R_n W_{n+1} - S_n V_{n+1} - \dots\dots - X_n Q_{n+1}.$$

If $\int_0^\alpha \frac{d\theta}{\sqrt{1 - x \sin^2\theta}} + \int_0^\beta \frac{d\theta}{\sqrt{1 - x\sin^2\theta}} = \int_0^\gamma \frac{d\theta}{\sqrt{1 - x \sin^2\theta}}$

then $\tan\frac{\gamma}{2} = \frac{\sin\alpha\sqrt{1 - x\sin^2\beta} + \sin\beta\sqrt{1 - x\sin^2\alpha}}{\cos\alpha + \cos\beta}$,

or $\cot\alpha\cot\beta = \frac{\cos\gamma}{\sin\alpha\sin\beta} + \sqrt{1 - x\sin^2\gamma}$

$\frac{\sqrt{x}}{2} = \frac{\sqrt{\sin\delta \sin(\delta-\alpha)\sin(\delta-\beta)\sin(\delta-\gamma)}}{\sin\alpha\sin\beta\sin\gamma}$ where $2\delta = \alpha + \beta + \gamma$.

Ex. 1. $1+\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\&c = 2\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)$

2. $1+\frac{2}{3^2}\cdot\frac{3}{4}+\frac{2\cdot4}{3\cdot5^2}\left(\frac{3}{4}\right)^2+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\left(\frac{3}{4}\right)^3+\&c$

$= \frac{\pi}{3\sqrt3}\log_e 3-\frac{10}{27}\pi^2+5\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{4^2}+\&c\right)$

3. $\frac{1}{2}+\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\&c$

$= -\frac{\pi}{6}\log_e(2+\sqrt3)+\frac{4}{3}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\&c\right)$

4. $1+\frac{2}{3^2}\cdot\frac{1}{2}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^2}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^3}+\&c$

$= -\frac{\pi}{2\sqrt2}\log_e(1+\sqrt2)-\frac{\pi^2}{4\sqrt2}+4\left(\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{13^2}+\&c\right)$

5. $\left(1-\frac{1}{2}\right)+\frac{2}{3^2}\left(1-\frac{3}{4^2}\right)+\frac{2\cdot4}{3\cdot5^2}\left(1-\frac{3}{4}\right)+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\left(1-\frac{5}{8}\right)+\&c$

$= \frac{\pi}{4}\log_e(2+\sqrt3)$

6. $\frac{2x}{1-x^2}-\frac{2}{3^2}\left(\frac{2x}{1-x^2}\right)^3+\frac{2\cdot4}{3\cdot5^2}\left(\frac{2x}{1-x^2}\right)^5-\&c$

$= 4\left(\frac{x}{1^2}+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\frac{x^7}{7^2}+\&c\right)$

7. $1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\&c = \frac{\pi^2}{8}-2\left\{\log_e(1+\sqrt2)\right\}^2$

8. $\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\&c$

$= \frac{\pi^2}{10}-\frac{3}{2}\left(\log\frac{\sqrt5+1}{2}\right)^2$

9. $\tan 2x-\frac{2}{3^2}\tan^3 2x+\frac{2\cdot4}{3\cdot5^2}\tan^5 2x-\&c$

$= 2\left(\tan x+\frac{\tan^3 x}{3^2}+\frac{\tan^5 x}{5^2}+\&c\right)$

10. $\left(\frac{x}{1+x}\right)+\frac{2}{3^2}\left(\frac{x}{1+x}\right)^2+\frac{2\cdot4}{3\cdot5^2}\left(\frac{x}{1+x}\right)^3+\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\left(\frac{x}{1+x}\right)^4+\&c$

$= x-\frac{2}{3}\left(1+\frac{1}{3}\right)x^2+\frac{2\cdot4}{3\cdot5}\left(1+\frac{1}{3}+\frac{1}{5}\right)x^3-\&c$

If $\dfrac{\tan\alpha}{\tan\beta} = \sqrt{1-x}$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{1-x\cos^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x\sin^2\theta}}.$$

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+x\sin\theta}} = \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x\sin^2\theta)}{\sqrt{1-x^2\sin^4\theta}}\,d\theta.$$

$$\left\{\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right\}^2 = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\,d\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\phi\sin^2\theta)}}$$

If $z = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c$

& $y = \pi\,\dfrac{1+\left(\frac{1}{2}\right)^2(1-x)+\&c}{1+\left(\frac{1}{2}\right)^2 x+\&c}$

then $\dfrac{1}{2} + \dfrac{\operatorname{sech} y}{1+n^2} + \dfrac{\operatorname{sech} 2y}{1+(2n)^2} + \dfrac{\operatorname{sech} 3y}{1+(3n)^2} + \&c$

$$= \cfrac{z}{2+\cfrac{(nz)^2 x}{2+\cfrac{(2nz)^2}{2+\cfrac{(3nz)^2 x}{2+\cfrac{(4nz)^2}{2+\&c}}}}}$$

$$\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\,d\phi = \cfrac{1}{n+\cfrac{x}{n+\cfrac{4}{n+\cfrac{9x}{n+\&c}}}}$$

If $\dfrac{\sin\alpha}{\sin\beta} = \sqrt{x}$, then

$$\int_0^\alpha \frac{d\theta}{\sqrt{x-\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

CHAPTER XII.

1 If x, y, z, u & n are positive integers, then

$$\frac{\lfloor x+y+n \;\lfloor z+u+n \;\lfloor u+x+n \;\lfloor x+y+z+n \;\lfloor y+z+u+n \;\lfloor z+u+x+n \;\lfloor u+x+y+n}{\lfloor n \;\lfloor x+y+n \;\lfloor y+z+n \;\lfloor z+u+n \;\lfloor u+x+n \;\lfloor x+z+n \;\lfloor y+u+n \;\lfloor x+y+z+u+n}$$

$$= n - (n+2)\frac{x}{1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}\cdot\frac{x+y+z+u+2n+1}{x+y+z+u+n}$$

$$+(n+4)\cdot\frac{x(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{u(u-1)}{(u+n+1)(u+n+2)}\cdot\frac{(x+y+z+u+2n+1)(x+y+z+u+2n+2)}{(x+y+z+u+n)(x+y+z+u+n+1)}$$

— &c

N.B. The above result is not true for all values of x, y, z, u and n. For example it is not true when x+y+z+u+n = [illegible] to the extraneous factor containing x, y, z and u [illegible] in each term. Unless we [illegible] all the quantities x, y, z & u [illegible] identities deduced from [illegible] get rid of this factor [illegible] won't be true for all values. The only way to get rid of this is to make it infinitely great. The solution of this theorem is evident from the result.

$$\Sigma\frac{1}{x+n}+\Sigma\frac{1}{y+n}+\Sigma\frac{1}{z+n}-\Sigma\frac{1}{x+y+n}-\Sigma\frac{1}{y+z+n}$$

$$-\Sigma\frac{1}{z+x+n}+\Sigma\frac{1}{x+y+z+n}-\Sigma\frac{1}{n}$$

$$=\left(1+\frac{1}{n+1}\right)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{x+y+z+n+1}{x+y+z+n}$$

$$+\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{(x+y+z+2n+1)(x+y+z+2n+2)}{(x+y+z+n)(x+y+z+n+1)}+\&c$$

This [illegible] for positive integral values

2. Substitute both sides in XII 1 [illegible]

[illegible] and then put [illegible]

3 If x, y, z and n are positive integers, then

If $\dfrac{\sin\beta}{\sin\alpha} = \dfrac{1+x}{1+x\sin^2\alpha}$,

then $$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

$$(1+2x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^3\frac{2+x}{1+2x}\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x\left(\frac{2+x}{1+2x}\right)^3\sin^2\theta}}$$

if $\dfrac{1+\sin\beta}{1-\sin\beta} = \dfrac{1+\sin\alpha}{1-\sin\alpha}\cdot\left(\dfrac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2$.

$$(1+x)^2\int_0^\alpha \frac{d\theta}{\sqrt{1-x^4\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-\left\{1-\left(\frac{1-x}{1+x}\right)^4\right\}\sin^2\theta}}$$

if $\dfrac{1+\sin\beta}{1-\sin\beta} = \dfrac{1+\sin\alpha}{1-\sin\alpha}\left(\dfrac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2\dfrac{1+x^2\sin\alpha}{1-x^2\sin\alpha}$,

If $\sin(2\beta-\alpha) = x\sin\alpha$

$$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 2\int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

If $\dfrac{\tan\alpha}{\tan\beta} = \sqrt{1+x}$

$$\Big\} - \int_0^\alpha \frac{d\theta}{\sqrt{1+x\cos 2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x^2\sin^4\theta}}$$

$$\frac{\lfloor x \lfloor x+y+n \lfloor y+z+n \lfloor z+x+n}{\lfloor x+n \lfloor y+n \lfloor z+n \lfloor x+y+z+n} = 1 + \frac{xyz}{\lfloor 2 (n+1)(x+y+z+n)}$$

$$+ \frac{x(x-1)\, y(y-1)\, z(z-1)}{\lfloor 3 (n+1)(n+2)(x+y+z+n)(x+y+z+n-1)} + \&c$$

Sol. Divide both sides in XII by n, write $-n+m$ for n and then make n infinitely great.

Ex. If x, y, z, and n are positive integers, then

$$\frac{(x+n)(y+n)(z+n)(x+y+z+n)}{(x+y+n)(y+z+n)(z+x+n)} = n +$$

$$(n+2)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{x+y+z+2n}{x+y+z+n-1} +$$

$$(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)} \times$$

$$\frac{(x+y+z+2n)(x+y+z+2n+1)}{(x+y+z+n-1)(x+y+z+n-2)} + \&c$$

Sol. Put $a = -1$ in XII.

Ex. If x is a positive integer shew that

1. $$\frac{(\lfloor x \lfloor 3x-2)^4}{(\lfloor x-1)^6 \lfloor 4x-3} = 1 - 3\left(\frac{x-1}{x+1}\right)^4 \frac{4x-1}{4x-3} + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^4 \frac{4x-1}{4x-3}\cdot\frac{4x}{4x-4}$$

$$- 7\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\cdot\frac{x-3}{x+3}\right)^4 \frac{4x-1}{4x-3}\cdot\frac{4x}{4x-4}\cdot\frac{4x+1}{4x-5} + \&c$$

2. $$\frac{3}{2} \cdot \frac{1}{\frac{x-1}{2}} - \frac{3}{2}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{\frac{2x-3}{2}}$$

$$= 1\left(\frac{x-1}{x+1}\right)^3 \frac{3x-1}{3x-3} + \frac{1}{2}\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3 \frac{3x-1}{3x-3}\cdot\frac{3x+1}{3x-5} + \&c$$

3. $$\left(\frac{x}{x-1}\right)^3 (3x-2) = 1 + 3\left(\frac{x-1}{x+1}\right)^3 \frac{3x-1}{3x-3} + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3 \frac{3x-1}{3x-3}\cdot\frac{3x}{3x-5}$$

$$+ \&c$$

4. $$\left(\frac{\lfloor x}{\lfloor x-1}\right)^3 \frac{1}{\lfloor 3x} = 1 + \left(\frac{x}{\lfloor 1}\right)^2 \frac{2x}{3x} + \left\{\frac{x(x-1)}{\lfloor 2}\right\}^2 \frac{x(x-1)}{3x(3x-1)} + \&c$$

5. $$\frac{8}{9}\left(\frac{\lfloor 3x}{\lfloor x}\right)^3 \frac{\lfloor x}{\lfloor 4x} = 1 + \frac{x}{\lfloor 1}\cdot\frac{x-1}{x+1}\cdot\frac{x}{4x-1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{(x-1)(x-2)}{(x+1)(x+2)}$$

$$\times \frac{x(x-1)}{(4x-1)(4x-2)} + \&c$$

$$x + (x+2)\frac{\alpha}{x-\alpha+1}\cdot\frac{\beta}{x-\beta+1} + (x+4)\frac{\alpha(\alpha+1)}{(x-\alpha+1)(x-\alpha+2)}$$

$$\times\frac{\beta(\beta+1)}{(x-\beta+1)(x-\beta+2)} + \&c \text{ to } k+1 \text{ terms}$$

$$= \frac{1}{x-\alpha-\beta}\left\{(x-\alpha)(x-\beta) - \frac{\underline{|x-\alpha}\,\underline{|x-\beta}}{\underline{|\alpha-1}\,\underline{|\beta-1}}\cdot\frac{\underline{|\alpha+k}\,\underline{|\beta+k}}{\underline{|x-\alpha+k}\,\underline{|x-\beta+k}}\right\}$$

If $\alpha+\beta+\gamma+1=k$, then

$$(k+1)\frac{\underline{|k}}{\underline{|0}}\cdot\frac{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}}{\underline{|k-\alpha}\,\underline{|k-\beta}\,\underline{|k-\gamma}} + (k+3)\frac{\underline{|k+1}}{\underline{|1}}\cdot\frac{\underline{|\alpha+1}\,\underline{|\beta+1}\,\underline{|\gamma+1}}{\underline{|k-\alpha+1}\,\underline{|k-\beta+1}\,\underline{|k-\gamma+1}}$$

$$+ (k+5)\frac{\underline{|k+2}}{\underline{|2}}\cdot\frac{\underline{|\alpha+2}\,\underline{|\beta+2}\,\underline{|\gamma+2}}{\underline{|k-\alpha+2}\,\underline{|k-\beta+2}\,\underline{|k-\gamma+2}} + \&c \text{ to } n \text{ terms}$$

$- 2\log_e 2n$ when n becomes infinite.

$$= -\Sigma\frac{1}{\alpha} - \Sigma\frac{1}{\beta} - \Sigma\frac{1}{\gamma} + 2C.$$

4. From all values of x, y, z and n,

$$n\cdot\frac{\underline{|x+n}\;\underline{|y+n}\;\underline{|z+n}\;\underline{|x+y+z+n}}{\underline{|n}\;\underline{|x+y+n}\;\underline{|y+z+n}\;\underline{|z+x+n}} = n - (n+2)\frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}$$

$$+ (n+4)\cdot\frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$- \&c.$

Sol. Make n infinite in XII 1.

6. $$\Sigma\frac{1}{x+n} + \Sigma\frac{1}{y+n} - \Sigma\frac{1}{x+y+n} - \Sigma\frac{1}{n} = \left(1 + \frac{1}{n+1}\right)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}$$

$$+ \left(\frac{1}{2} + \frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

Sol. Subtract both sides in XII 5 from n, divide both sides by z and then put $z=0$.

7. $$\frac{(x+n)(y+n)}{x+y+n} = n + (n+2)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1} +$$

$$(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

Sol. Put $z = -1$ in XII 5.

8. $$n\,\frac{\underline{|x+y}}{\underline{|x}\;\underline{|y}}\cdot\frac{\underline{|x+n}\;\underline{|y+n}}{\underline{|n}\;\underline{|x+y+n}} = n + (n+2)\frac{n^2}{(\underline{|1})^2}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}$$

$$+ (n+4)\frac{n^2(n+1)^2}{(\underline{|2})^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

Sol. Write $-n$ for z in XII 5.

9. $$\frac{\underline{|x+n}\;\underline{|y+n}}{\underline{|n}\;\underline{|x+y+n}}\cdot\frac{\underline{|\tfrac{n}{2}}\;\underline{|x+y+\tfrac{n}{2}}}{\underline{|x+\tfrac{n}{2}}\;\underline{|y+\tfrac{n}{2}}} = 1 + \frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}$$

$$+ \frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

Sol. Put $z = -\frac{n}{2}$ in XII 5.
It is also valid ... as the isolated factors n, $n+2$, $n+4$ disappear.

10. $$n\,\frac{\underline{|x+n}\;\underline{|y+n}}{\underline{|n}\;\underline{|x+y+n}}\cdot\frac{\underline{|\tfrac{n-2}{2}}\;\underline{|x+y+\tfrac{n-2}{2}}}{\underline{|x+\tfrac{n-2}{2}}\;\underline{|y+\tfrac{n-2}{2}}} = n + (n+2)\frac{n}{\underline{|1}}\cdot\frac{xy}{(x+n+1)(y+n+1)}$$

$$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 2\int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

if $(1+x)\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\beta} = \sqrt{1-x^2\sin^2\alpha} + x\cos\alpha$.

$$\int_0^\alpha \frac{d\theta}{\sqrt{1-x\sin^2\theta}} + \int_0^\beta \frac{d\theta}{\sqrt{1-x\sin^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

~~if $(1-x\sin^2\alpha)(1-x\sin^2\beta) = 1-x$.~~

if $\cot\alpha \cdot \cot\beta = \sqrt{1-x}$

$$\int_0^\alpha \frac{d\theta}{\sqrt{1-x\sin^2\theta}} = 2\int_0^\beta \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

if $\frac{\tan\frac{\alpha}{2}}{\tan\beta} = \sqrt{1-x\sin^2\beta}$.

$$\int_0^{\sin^{-1}\sqrt[8]{\frac{\beta}{\alpha^3}}} \frac{d\theta}{\sqrt{1-\alpha\sin^2\theta}} = \frac{\pi}{8}\left\{1+\left(\tfrac{1}{4}\right)^2\alpha + \right\}$$

where $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1$.

$$+ (n+4)\frac{n(n+1)}{\underline{|2}} \cdot \frac{x(x-1)}{(x+n+1)(x+n+2)} \cdot \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

11. $$\frac{\underline{|x+n}\,\underline{|y+n}}{\underline{|x+y+n}} = n - (n+2)\frac{n}{1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1} +$$

$$(n+4)\frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)} - \&c$$

12. $$\left\{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \&c\right\} - \left\{\frac{1}{(x+n+1)^2} + \frac{1}{(x+n+2)^2} + \&c\right\}$$

$$= \left(1 + \frac{1}{n+1}\right)\frac{1}{n+1}\cdot\frac{x}{x+n+1} - \left(\frac{1}{2} + \frac{1}{n+2}\right)\frac{\underline{|1}}{(n+1)(n+2)}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}$$

$$+ \left(\frac{1}{3} + \frac{1}{n+3}\right)\frac{\underline{|2}}{(n+1)(n+2)(n+3)}\cdot\frac{x(x-1)(x-2)}{(x+n+1)(x+n+2)(x+n+3)} - \&c$$

Sol. Divide both sides in Art 6 by y and then put $y=0$.

13. $$\frac{(n-1)(x+n)}{x+n-1} = n - (n+2)\frac{\underline{|1}}{n}\cdot\frac{x}{x+n+1} +$$

$$(n+4)\frac{\underline{|2}}{n(n+1)}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$$

14. $$0 = n - (n+2)\frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1} + (n+4)\frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$$

15. $$\frac{x+n}{2x+n} = 1 - \frac{x}{x+n+1} + \frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$$

16. $$\frac{(n-1)(x+n)}{2x+n-1} = n - (n+2)\frac{x}{x+n+1} + (n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$$

17. $$x+n = n + (n+2)\frac{x}{x+n+1} + (n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$$

18. $$\frac{\underline{|x+n}\,\underline{|x-n}}{(\underline{|x})^2}\cdot\frac{\sin \pi n}{\pi} = n - (n+2)\frac{n^2}{(\underline{|1})^2}\cdot\frac{x}{x+n+1}$$

$$+ (n+4)\frac{n^2(n+1)^2}{(\underline{|2})^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}$$

19. $$\frac{\underline{|x+n}}{\underline{|x}\,\underline{|n}} \cdot \frac{\underline{|x}\,\underline{|x-x}}{\underline{|x}\,\underline{|x}} = 1 - \frac{n^2}{(\underline{|1})^2}\cdot\frac{x}{x+n+1}$$

$$+ \frac{n^2(n+1)^2}{(\underline{|2})^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$$

20. $$n\cdot\frac{\underline{|x-\frac{n+1}{2}}}{\underline{|x}\,\underline{|-\frac{n+1}{2}}}\cdot\frac{\underline{|x+n}\,\underline{|\frac{n}{2}}}{\underline{|x+\frac{n}{2}}\,\underline{|n}} = n - (n+2)\frac{n^2}{(\underline{|1})^2}\cdot\frac{x}{x+n+1}$$

$$+ (n+4)\frac{n^2(n+1)^2}{(\underline{|2})^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}$$

$$\frac{1}{a+\frac{x^2}{a}} - \frac{2a}{\lfloor 1}\cdot\frac{1}{a+1+\frac{x^2}{a+1}} + \frac{2a(2a+1)}{\lfloor 2}\cdot\frac{1}{a+2+\frac{x^2}{a+2}} - \&c$$

$$= \frac{2\left(\lfloor a-1\right)^2 / \lfloor 2a-1}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\&c}$$

$$\int_0^\infty \frac{\phi(xi)+\phi(-xi)}{1+x^2}\,dx = \pi\,\phi(1).$$

If $\int_0^\infty \phi(x)\sin 2nx\,dx = \psi(n)$

then $\alpha\left\{\phi(\alpha)\sin 2n\alpha + \phi(2\alpha)\sin 4n\alpha + \&c\right\}$

$$= \psi(n) - \psi(\beta-n) + \psi(\beta+n) - \psi(2\beta-n) + \&c$$

with $\alpha\beta = \pi$ & n lying between 0 & β.

$(n-1)$ th $= \sqrt[n]{\alpha\beta} + \sqrt[n]{(1-\alpha)(1-\beta)} = 1$

then $(n-1)^2$ th $= \left(\sqrt[n]{\alpha} - \sqrt[n]{\beta}\right)^n + \left(\sqrt[n]{1-\beta} - \sqrt[n]{1-\alpha}\right)^n$

$$= \left\{\sqrt[n]{\alpha(1-\beta)} - \sqrt[n]{\beta(1-\alpha)}\right\}^n.$$

21. $\frac{x\,\underline{|x+n}}{\underline{|2x}} = x + (x+2)\frac{n^2}{(\underline{|1})^2}\cdot\frac{x}{x+n+1} + (n+4)\frac{n^2(n+1)^2}{(\underline{|2})^2}\frac{x(x-1)}{(x+n+1)(x+n+2)}$

$+ \&c$

22. $\frac{\underline{|x+n}\,(\underline{|x})^2}{\underline{|2}\,(\underline{|x+n})^2} = \frac{1}{n} - \frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1}\cdot\frac{1}{n+2}$

$+ \frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{1}{n+4} - \&c$

23. $\frac{\underline{|2x}\,\underline{|x+n}}{\underline{|x}\,\underline{|2x+n}} = 1 - \frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$

24. $\frac{\underline{|x+n}\,\underline{|\frac{x}{2}}}{\underline{|x}\,\underline{|x+\frac{n}{2}}} = 1 + \frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

25. $\frac{\underline{|x+n}\,\underline{|\frac{n-1}{2}}}{\underline{|n-1}\,\underline{|x+\frac{n-1}{2}}} = n + (n+2)\frac{n}{\underline{|1}}\cdot\frac{x}{x+n+1} +$

$(n+4)\frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

26. $\frac{(n-1)^2}{n-2} = n + (n+2)\frac{1^2}{n^2} + (n+4)\frac{1^2\cdot 2^2}{n^2(n+1)^2} + \&c$

27. $\frac{n-1}{n-2} = 1 + \frac{1}{n} + \frac{1\cdot 2}{n(n+1)} + \frac{1\cdot 2\cdot 3}{n(n+1)(n+2)} + \&c$

28. $\frac{(n-1)^2}{n-3} = n + (n+2)\frac{1}{n} + (n+4)\frac{1\cdot 2}{n(n+1)} + \&c$

29. $n-1 = n - (n+2)\frac{1}{n} + (n+4)\frac{1\cdot 2}{n(n+1)} - \&c$

30. $\frac{(\underline{|n})^2}{\underline{|2n}}\cdot\frac{\sin\pi n\,\sin\pi n}{\pi^2 n} = n + (n+2)\frac{n^4}{(\underline{|1})^4} + (n+4)\frac{n^4(n+1)^4}{(\underline{|2})^4} + \&c$

31. $\frac{(\underline{|\frac{n}{2}})^3\,\sin\pi n\,\sin\frac{\pi n}{2}}{\pi^2 n(1+2\cos\pi n)\,\underline{|\frac{3n}{2}}} = 1 + \frac{n^3}{(\underline{|1})^3} + \frac{n^3(n+1)^3}{(\underline{|2})^3} + \&c$

32. $\frac{\underline{|\frac{n-1}{2}}\,\underline{|-\frac{3n+1}{2}}}{(\underline{|\frac{n-1}{2}})^2}\cdot\frac{\sin\pi n}{\pi} = n + (n+2)\frac{n^3}{(\underline{|1})^3} + (n+4)\frac{n^3(n+1)^3}{(\underline{|2})^3} + \&c$

33. $\frac{\sin\pi n}{\pi} = n - (n+2)\frac{n^3}{(\underline{|1})^3} + (n+4)\frac{n^3(n+1)^3}{(\underline{|2})^3} - \&c$

$(\underline{|\frac{n}{2}})^4$

$= \frac{1}{n} + \frac{1}{n+2}\frac{n^2}{(\underline{|1})^2} + \frac{1}{n+4}\frac{n^2(n+1)^2}{(\underline{|2})^2} + \&c$

$\pi^{2}\ldots$

$$\int_0^\infty \frac{\cos 2nx\, dx}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\cdots} = \frac{\sqrt{\pi}}{2}\cdot\frac{\underline{|a-\tfrac{1}{2}}}{\underline{|a-1}}\operatorname{sech}^{2a} n.$$

$$\int_0^\infty \frac{1}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\cdots}\cdot\frac{dx}{\left\{1+\left(\frac{x}{b}\right)^2\right\}\left\{1+\left(\frac{x}{b+1}\right)^2\right\}\cdots}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\underline{|a-\tfrac{1}{2}}\;\underline{|b-\tfrac{1}{2}}\;\underline{|a+b-1}}{\underline{|a-1}\;\underline{|b-1}\;\underline{|a+b-\tfrac{1}{2}}}.$$

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)\left(1+\frac{x^2}{d^2}\right)}$$

$$= \frac{\pi}{6}\cdot\frac{abcd\{(a+b+c+d)^3-(a^3+b^3+c^3+d^3)\}}{(a+b)(b+c)(c+a)(a+d)(b+d)(c+d)}.$$

If α, β, γ, & δ are the roots of the equation $x^4-px^3+qx^2-rx+s=0$, then

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^2}{\alpha^2}\right)\left(1+\frac{x^2}{\beta^2}\right)\left(1+\frac{x^2}{\gamma^2}\right)\left(1+\frac{x^2}{\delta^2}\right)} = \frac{\pi}{2}\cdot\frac{s}{r-\frac{ps}{q-\frac{r}{p}}}.$$

If $\int_0^\infty \phi(a,x)\cos nx\, dx = \psi(a,n)$, then

$$\int_0^\infty \phi(a,x)\,\phi(b,x)\,dx = \int_0^\infty \psi(a,x)\,\psi(b,\ )\,dx$$

$$\frac{\pi}{2}\int_0^\infty \phi(a,x)$$

$\frac{\pi(|x)^2}{2x|2x \sin\frac{\pi x}{2}} = \frac{1}{x^2} + \frac{[illegible]}{(x+1)^2} + \frac{[illegible]}{(x+2)^2} + \&c$

36. $\sum\frac{1}{x+n} - \sum\frac{1}{x} = \left(1+\frac{1}{n+1}\right)\frac{x}{x+n+1} + \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

37. $\frac{1}{x} + \sum\frac{1}{n} - \sum\frac{1}{x+n} = \left(1+\frac{1}{n+1}\right)\frac{n}{1}\cdot\frac{x}{x+n+1} + \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{n(n+1)}{1\cdot 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

38. $2n^2\left\{\frac{1}{n^3}+\frac{1}{(n+1)^3}+\frac{1}{(n+2)^3}+\&c\right\} = \left(1+\frac{1}{n}\right) + \left(\frac{1}{2}+\frac{1}{n+1}\right)\left(\frac{1}{n+1}\right)^2 + \left(\frac{1}{3}+\frac{1}{n+2}\right)\left(\frac{1}{n+1}\cdot\frac{2}{n+2}\right)^2 + \&c$

39. $\left\{\frac{1}{(1+\frac{n}{2})^2}+\frac{1}{(2+\frac{n}{2})^2}+\&c\right\} - \left\{\frac{1}{(1+n)^2}+\frac{1}{(2+n)^2}+\&c\right\} = \left(1-\frac{1}{n+1}\right)\frac{1}{n+1} + \left(\frac{1}{2}-\frac{1}{n+2}\right)\frac{|2}{(n+1)(n+2)} + \&c$

40. $\sum\frac{1}{x} + \sum\frac{1}{n} - [illegible] = \left(1+\frac{1}{n+1}\right)\frac{x^2}{(|1)^2} + \left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x^2(x-1)^2}{(|2)^2} + \&c$

Ex. 1. $\frac{(|x)^3\,|3x-2}{(|2x-1)^3} = 1 - 3\left(\frac{x-1}{x+1}\right)^3 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3 - \&c$

2. $\frac{x^2}{2x-1} = 1 + 3\left(\frac{x-1}{x+1}\right)^2 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 + \&c$

3. $\frac{(|x)^4\,|4x}{(|2x)^4}\cdot\frac{2x}{4x-1} = 1 + \left(\frac{x-1}{x+1}\right)^2 + \left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 + \&c$

4. $\frac{(|x)^2}{|2x-1} = 1 - 3\left(\frac{x-1}{x+1}\right)^2 + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2 - \&c$

5. $x = 1 + 3\,\frac{x-1}{x+1} + 5\,\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + \&c$

6. $\frac{2^{2x-1}(|x)^2}{|2x} = 1 + \frac{x-1}{x+1} + \frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + \&c$

7. $\frac{1}{2x-1} = 1 - \frac{x-1}{x+1} + \frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \&c$

8. $[illegible] = 1 - 3\,\frac{x-1}{x+1} + 5\,\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \&c$

9. $\frac{1}{2}\sum\frac{1}{x} + \frac{(|x-1\,|x-1)^2}{|2x-2} = 1 + \frac{1}{2}\cdot\frac{x-1}{x+1} + \frac{1}{3}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + [illegible]$

10. $[illegible] = 1 + [illegible]$

$$\int_0^\infty x^{n-1}\left\{\phi(0)-\frac{x}{\lfloor\underline{1}}\phi(1)+\frac{x^2}{\lfloor\underline{2}}\phi(2)-\frac{x^3}{\lfloor\underline{3}}\phi(3)+\&c\right\}dx$$

$$=\lfloor\underline{n-1}\;\phi(-n).$$

$$\lfloor\underline{n}\int_0^\infty \frac{\lfloor\underline{x+p-1}}{x^m\lfloor\underline{x+p+n}}\,dx$$

$$=\frac{\pi}{\sin\pi m}\left\{\frac{1}{p^m}-\frac{n}{\lfloor\underline{1}}\cdot\frac{1}{(p+1)^m}+\frac{n(n-1)}{\lfloor\underline{2}}\cdot\frac{1}{(p+2)^m}-\&c\right\}$$

$$\int_0^\infty \frac{1}{x^{n+1}}\cdot\frac{1-px}{1+x}\cdot\frac{1-prx}{1+rx}\cdot\frac{1-pr^2x}{1+r^2x}\cdot\frac{1-pr^3x}{1+r^3x}\,\&c\;dx$$

$$=\text{[struck-through term]}\cdot\frac{1+p}{1+pr^n}\cdot\frac{1+pr}{1+pr^{n+1}}\cdot\frac{1+pr^2}{1+pr^{n+2}}\,\&c\times-\frac{\pi}{\sin\pi n}$$

$$\times\frac{1-r^{n+1}}{1-r}\cdot\frac{1-r^{n+2}}{1-r^2}\cdot\frac{1-r^{n+3}}{1-r^3}\,\&c$$

$$\int_0^\infty \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+2}\right)^2},\&c\;dx$$

$$=\frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor\underline{a-\tfrac{1}{2}}\,\lfloor\underline{b}}{\lfloor\underline{a-1}\,\lfloor\underline{b-\tfrac{1}{2}}}\cdot\frac{\lfloor\underline{b-a-\tfrac{1}{2}}}{\lfloor\underline{b-a}}.$$

Subtract

$$1 - \frac{1}{2^2}\cdot\frac{x-1}{x+1} + \frac{1}{3^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \frac{1}{4^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\cdot\frac{x-3}{x+3} + \&c$$

$$x(4x-3) = 1^3 + 3^3\frac{x-1}{x+1} + 5^3\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} + \&c$$

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot3}{2\cdot4}\right)^3 - 13\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3 + \&c$$

$$\frac{\pi^2}{8}\left\{1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1\cdot5}{4\cdot8}\right)^4 + 25\left(\frac{1\cdot5\cdot9}{4\cdot8\cdot12}\right)^4 + \&c\right\}$$

$$= 1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1\cdot3}{2\cdot4}\right)^3 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3 + \&c$$

$$1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1\cdot3}{2\cdot4}\right)^3 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3 + \&c = 2\left\{1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 + \&c\right\}^2$$

$$1 + \frac{1}{5}\left(\frac{1}{2}\right)^2 + \frac{1}{9}\left(\frac{1\cdot3}{2\cdot4}\right)^2 + \frac{1}{13}\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 + \&c$$

$$= \frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 + \&c\right\}$$

$$\frac{1}{1^2} + \left(\frac{1}{2}\right)^2\frac{1}{5^2} + \left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{1}{9^2} + \&c = \frac{\pi^2}{32}\left\{1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1\cdot5}{4\cdot8}\right)^4 + \&c\right\}$$

11. $$\frac{\lfloor n \; \lfloor x+y+n}{\lfloor x+n \; \lfloor y+n} = 1 + \frac{x}{1}\cdot\frac{y}{n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{(n+1)(n+2)} + \&c$$

Sol. Write $-n+m$ for x in XII and make n infinite

... the coeff.ts of A^n in $(1+A)^{y+n}\left(1+\frac{1}{A}\right)^x = \frac{(1+A)^{x+y+n}}{A^x}$

$$\Sigma\frac{1}{a-?} - \Sigma\frac{1}{a-b-1} = \frac{b}{a} + \frac{b(b+1)}{a(a+1)}\cdot\frac{1}{2} + \frac{b(b+1)(b+2)}{a(a+1)(a+2)}\cdot\frac{1}{3} + \&c$$

Sol. Subtract 1 from both sides in XII 11, ...

$$\frac{\lfloor n \; \lfloor x}{n+x} = \frac{1}{n} - \frac{x}{1}\cdot\frac{1}{n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{1}{n+2} - \&c$$

Sol. Write $-m$ for y in XII 11.

... &c

$$\int_0^\infty \frac{\lfloor x+\alpha}{\lfloor x+\beta} \left(\Sigma \frac{1}{x+\beta} - \Sigma \frac{1}{x+\alpha} \right) dx$$

$$= \frac{\lfloor \alpha}{\lfloor \beta}.$$ *Integration very simple*

$$\int_0^{\frac{\pi}{2}} \cos^m x \cos nx \, dx = \frac{\pi \lfloor m}{2^{m+1} \lfloor \frac{m+n}{2} \lfloor \frac{m-n}{2}}.$$

$$\int_0^\infty \frac{\sin^n x}{x^p} dx = \frac{1}{\lfloor p-1} \int_0^\infty \int_0^\infty z^{p-1} e^{-zx} \sin^n x \, dz \, dx$$

$$\int_0^\infty e^{-ax} \sin^{2n+1} x \, dx = \frac{\lfloor 2n+1}{(a^2+1^2)(a^2+3^2) \cdots (a^2+\overline{2n+1}^2)}$$

$$\int_0^\infty e^{-ax} \sin^{2n} x \, dx = \frac{\lfloor 2n}{a(a^2+2^2)(a^2+4^2) \cdots (a^2+\overline{2n}^2)}$$

change x to xi then we get $e^x - e^{-x}$

$\frac{\pi}{\sin \pi x} = \frac{1}{x} + \frac{x}{1}\cdot\frac{1}{x+1} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{1}{x+2} + \&c$

$3.\ \frac{(2^x\,\underline{|x})^2}{x\,\underline{|2x}} = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x+1} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{x+2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{x+3} + \&c$

$4.\ \frac{\pi}{\sin\pi x}\cdot\frac{\underline{|2x}}{(2^x\,\underline{|x})^2(1-2x)} = 1 + \frac{x}{1}\cdot\frac{1}{3} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{1}{5} + \&c$

$5.\ \frac{\underline{|x}\,\underline{|n-1}}{\underline{|x+n}}\left(\Sigma\frac{1}{x+n} - \Sigma\frac{1}{n-1}\right) = \frac{1}{n^2} - \frac{x}{1}\cdot\frac{1}{(n+1)^2} + \frac{x(x-1)}{\underline{|2}}\cdot\frac{1}{(n+2)^2}$

$6.\ \frac{(2^x\,\underline{|x})^2}{x\,\underline{|2x}}\left(2\Sigma\frac{1}{2x} - 2\Sigma\frac{1}{x} + \frac{1}{x} - 2\log_e 2\right)$

$= \frac{1}{x^2} + \frac{1}{2}\cdot\frac{1}{(x+1)^2} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{(x+2)^2} + \&c$

$7.\ \frac{\pi}{\tan\pi x}\cdot\frac{\underline{|2x}}{(1-2x)(2^x\,\underline{|x})^2}\left(\Sigma\frac{1}{2x} - \frac{1}{2}\Sigma\frac{1}{x} + \frac{1}{1-2x} - \frac{\pi}{2}\tan\pi x\right)$

$= \frac{1}{1^2} + \frac{x}{1}\cdot\frac{1}{3^2} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{1}{5^2} + \&c$

$8.\ -\frac{\pi}{\sin\pi x}\,\Sigma\frac{1}{x} = \frac{1}{x^2} + \frac{x}{1}\cdot\frac{1}{(x+1)^2} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{1}{(x+2)^2} + \&c$

$14.\ a^n = \{a^n - (b+1)^n\} + \{(a+1)^n - (b+2)^n\}\left(\frac{b+1}{a+1}\right)^n +$

$\{(a+2)^n - (b+3)^n\}\left(\frac{b+1}{a+1}\right)^n\left(\frac{b+2}{a+2}\right)^n + \&c$

Sol. $a^n = \{a^n - (b+1)^n\} + (b+1)^n = \{a^n - (b+1)^n\} + (a+1)^n\left(\frac{b+1}{a+1}\right)^n$

$= \&c\ \&c$

Cor. $\frac{b}{a-b-1} = \frac{b}{a} + \frac{b(b+1)}{a(a+1)} + \frac{b(b+1)(b+2)}{a(a+1)(a+2)} + \&c$

$= (a+b+1)\frac{b^2}{a^2} + (a+b+1)\frac{b^2(b+1)^2}{a^2(a+1)^2} + \&c$

15. If $e^{A_1 x + A_2\frac{x^2}{2} + A_3\frac{x^3}{3} + \&c}$

$= P_0 + P_1 x + P_2 x^2 + \&c$, then

$nP_n = P_{n-1}A_1 + P_{n-2}A_2 + P_{n-3}A_3 + \&c$ to n terms

Sol. ... by taking logarithms of both sides and then differentiating

$\frac{ab}{cd} = k$. Then

$f(a,b)f(c,d) + f(-a,-b)f(-c,-d)\}$

$f(ac,bd) + ad\,f\left(ack, \frac{bd}{k}\right) + bc\,f\left(bdk, \frac{ac}{k}\right)$

$+ (ad)^3 bc\,f\left(ack^2, \frac{bd}{k^2}\right) + (bc)^3 ad\,f\left(bdk^2, \frac{ac}{k^2}\right)$

$+ (ad)^6 (bc)^3 f\left(ack^3, \frac{bd}{k^3}\right) + (bc)^6 (ad)^3 f\left(bdk^3, \frac{ac}{k^3}\right)$

$+ \&c$

$\frac{1}{2}\{f(a,b)f(c,d) - f(-a,-b)f(-c,-d)\}$

$= a\,f\left(\frac{c}{a}, \frac{a}{c}.abcd\right) + d\,f\left(\frac{b}{d}, \frac{d}{b}.abcd\right)$

$+ a^3bc\,f\left(\frac{c}{ak}, \frac{ak}{c}.abcd\right) + d^3bc\,f\left(\frac{bk}{d}, \frac{d}{bk}.abcd\right)$

$+ a^6d(bc)^3 f\left(\frac{c}{ak^2}, \frac{ak^2}{c}.abcd\right) + ad^6(bc)^3 f\left(\frac{bk^2}{d}, \frac{d}{bk^2}.abcd\right)$

$+ \&c$

of the products of $a_1, a_2, a_3 \ldots a_n$ taken r at a time

then $rP_r = P_{r-1}S_1 - P_{r-2}S_2 + P_{r-3}S_3 - P_{r-4}S_4 +$ &c and $P_0 = 1$

46. $\frac{1}{x^{n+1}} - \frac{x}{|1} \cdot \frac{1}{(x+1)^{n+1}} + \frac{x(x-1)}{|2} \cdot \frac{1}{(x+2)^{n+1}}$ &c

$= \frac{|x \; |n}{n \; |x+n} \phi(n)$ where $\phi(1) = 1$ and

$n\phi(n) = S_1\phi(n-1) + S_2\phi(n-2) + S_3\phi(n-3) +$ &c to n terms

where $S_r = \frac{1}{x^r} - \frac{1}{(x+n+1)^r} + \frac{1}{(n+1)^r} - \frac{1}{(x+n+2)^r} +$ &c

Cor 1. $1 + \frac{1}{2} \cdot \frac{1}{3^{n+1}} + \frac{1.3}{2.4} \cdot \frac{1}{5^{n+1}} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7^{n+1}} +$ &c $= \frac{\pi}{2}\phi(n)$

where $\phi(1) = 1$ and $n\phi(n) = S_1\phi(n-1) + S_2\phi(n-2) + S_3\phi(n-3) +$ &c

to n terms where $S_r = \frac{1}{1^r} - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} +$ &c

Cor 2. $\frac{1}{2^{n+1}} + \frac{1}{2} \cdot \frac{1}{4^{n+1}} + \frac{1.3}{2.4} \cdot \frac{1}{6^{n+1}} +$ &c $= \phi(n)$ where

$\phi(1) = 1$ and $n\phi(n) = S_1\phi(n-1) + S_2\phi(n-2) +$ &c where

$S_r = \frac{1}{2^r} - \frac{1}{3^r} + \frac{1}{4^r} - \frac{1}{5^r} +$ &c.

Sol. Write n - P for n in XII 43 and equate the coeffts of p

Ex. $1 + \frac{1}{2} \cdot \frac{1}{3^3} + \frac{1.3}{2.4} \cdot \frac{1}{5^3} +$ &c $= \frac{\pi^3}{48} + \frac{\pi}{8}(\log 2)^2$

47. $\frac{1}{(x+1)^n} + \frac{1+\frac{1}{2}}{(x+2)^n} + \frac{1+\frac{1}{2}+\frac{1}{3}}{(x+3)^n} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{(x+4)^n} +$ &c $= \frac{n}{2}S_{n+1}$

$-(S_1S_n + S_2S_{n-1} + S_3S_{n-2} +$ &c the last term being $S_{\frac{n}{2}}S_{\frac{n}{2}+1}$

or $\frac{1}{2}S_{\frac{n+1}{2}}S_{\frac{n+1}{2}}$ according as n is even or odd) where

$S_n = \frac{1}{x^n} + \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} +$ &c and $S_1 = \ldots \frac{1}{x}$ [illegible]

$\frac{1}{n}\left(1 + \ldots \frac{1}{n+1}\right) + \left(1+\frac{1}{2}\right)\left(\frac{1}{3} \ldots \frac{1}{n+3}\right) + \left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{4} \ldots\right)$ [illegible]

$= \frac{1}{2}\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}\right)^2 + \left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots\frac{1}{n^2}\right)\right\}$

in this probability [illegible]

[illegible]

If $$P = \frac{l}{x+}\,\frac{1^2-n^2}{x+}\,\frac{2^2-l^2}{x+}\,\frac{3^2-n^2}{x+}\,\frac{4^2-l^2}{x+\&c}$$

then $$\frac{1-P}{1+P} = \frac{\left\lfloor \frac{x+l+n-3}{4} \right. \left\lfloor \frac{x+l-n-3}{4} \right. \left\lfloor \frac{x-l+n-1}{4} \right. \left\lfloor \frac{x-l-n-1}{4} \right.}{\left\lfloor \frac{x-l+n-3}{4} \right. \left\lfloor \frac{x-l-n-3}{4} \right. \left\lfloor \frac{x+l+n-1}{4} \right. \left\lfloor \frac{x+l-n-1}{4} \right.}$$

$$\cdot\,\frac{\left\lfloor \frac{x+l+n-3}{4} \right. \left\lfloor \frac{x-l+n-3}{4} \right. \left\lfloor \frac{x+l-n-3}{4} \right. \left\lfloor \frac{x-l-n-3}{4} \right.}{\left\lfloor \frac{x+l+n-1}{4} \right. \left\lfloor \frac{x-l+n-1}{4} \right. \left\lfloor \frac{x+l-n-1}{4} \right. \left\lfloor \frac{x-l-n-1}{4} \right.}$$

$$= \frac{8}{\frac{x^2-l^2+n^2-1}{2}+}\,\frac{1^2-n^2}{1+}\,\frac{1^2-l^2}{x^2-1+}\,\frac{3^2-n^2}{1+}\,\frac{3^2-l^2}{x^2-1+\&c}$$

$$\left\{\frac{1}{(x-n+1)^2}+\frac{1}{(x-n+3)^2}+\frac{1}{(x-n+5)^2}+\&c\right\}$$

$$-\left\{\frac{1}{(x+n+1)^2}+\frac{1}{(x+n+3)^2}+\frac{1}{(x+n+5)^2}+\&c\right\}$$

$$= \frac{n}{x^2-1+n^2+}\,\frac{2(1^2-n^2)}{1+}\,\frac{2}{3(x^2-1)+n^2+}\,\frac{4(2^2-n^2)}{1+\&c}$$

$$= \frac{n}{x^2-n^2+1-}\,\frac{4(1^2-n^2)}{3(x^2-n^2+5)-\&c}$$

CHAPTER XIII

1. If $P_1 - nA_1 + \frac{n(n-1)}{\lfloor 2} A_2 - \frac{n(n-1)(n-2)}{\lfloor 3} A_3 + \&c = P_n$, then

$$A_0 - nP_1 + \frac{n(n-1)}{\lfloor 2} P_2 - \frac{n(n-1)(n-2)}{\lfloor 3} P_3 + \&c = A_n.$$

2. $\frac{A_0}{x^n} + \frac{n}{\lfloor 1} \cdot \frac{A_1}{x^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{x^{n+2}} + \&c$

$$= \frac{A_0}{(x+h)^n} + \frac{n}{\lfloor 1} \cdot \frac{A_1 + hA_0}{(x+h)^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2 + 2hA_1 + h^2A_0}{(x+h)^{n+2}} + \&c$$

3. If $\frac{A_0}{x^n} + \frac{n}{\lfloor 1} \frac{A_1}{x^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{x^{n+2}} + \&c$

$$= \frac{A_0}{(x-1)^n} - \frac{n}{\lfloor 1} \cdot \frac{A_1}{(x-1)^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{(x-1)^{n+2}} - \&c, \text{ then}$$

(a) $$e^x = \frac{A_0 + \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 + \frac{x^3}{\lfloor 3} A_3 + \&c}{A_0 - \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 - \frac{x^3}{\lfloor 3} A_3 + \&c}$$

Sol. Multiplying both sides in XIII 3 by x^n and making n infinite ... such that $\frac{n}{x} = y$

(b). $$\frac{1}{\{\phi(x)\}^n}\left[A_0 + A_1 \frac{n}{\lfloor 1}\left\{\frac{\phi(x) - \phi(-x)}{\phi(x)}\right\} + A_2 \frac{n(n+1)}{\lfloor 2}\left\{\frac{\phi(x) - \phi(-x)}{\phi(x)}\right\}^2 + \&c\right]$$

is always an even function of x whatever be $\phi(x)$

Sol. Writing in XIII 3 $\frac{\phi(x)}{\phi(x) - \phi(-x)}$ for x, we have

$$\frac{A_0}{\{\phi(x)\}^n} + A_1 \frac{n}{\lfloor 1} \frac{\phi(x) - \phi(-x)}{\{\phi(x)\}^{n+1}} + A_2 \frac{n(n+1)}{\lfloor 2} \frac{\{\phi(x) - \phi(-x)\}^2}{\{\phi(x)\}^{n+2}} + \&c$$

$$= \frac{A_0}{\{\phi(-x)\}^n} + A_1 \frac{n}{\lfloor 1} \frac{\phi(-x) - \phi(x)}{\{\phi(-x)\}^{n+1}} + A_2 \frac{n(n+1)}{\lfloor 2} \frac{\{\phi(-x) - \phi(x)\}^2}{\{\phi(-x)\}^{n+2}} \ldots$$

∴ Each of these is an even function of x.

(c) If n is even, the value of A_{n+1} depends upon the values of A_n ... But ... any value we choose ...

$$\frac{A_{n-1}}{n} = \frac{n-1}{\lfloor 2} (2^2-1) B_1 A_{n-2} - \frac{(n-1)(n-2)(n-3)}{\lfloor 4} (2^4-1) B_2 A_{n-4} + \ldots$$

... $(2^6-1) B_3 A_{n-6}$ ...

$$\phi(0) + \frac{m}{n}\cdot\frac{\phi'(0)}{\lfloor 1} + \frac{m(m+1)}{n(n+1)}\frac{\phi''(0)}{\lfloor 2} + \&c.$$

$$= \phi(1) + \frac{m-n}{n}\frac{\phi'(1)}{\lfloor 1} + \frac{(m-n)(m-n-1)}{n(n+1)}\cdot\frac{\phi''(1)}{\lfloor 2} + \&c$$

$$V = \cfrac{2lmn}{y+p-2ml^2 + \cfrac{2(1-m)(1^2-n^2)}{1 + \cfrac{2(1+m)(1^2-l^2)}{3y+p + \cfrac{2(2-m)(2^2-n^2)}{1 + \cfrac{2(2+m)(2^2-l^2)}{5y+p + \&c\ \&c}}}}}$$

where $y = x^2 - (1-m)^2$ & $p = (n^2-l^2)(1-2m)$.

If $\phi(x,y) = x + \cfrac{(1+y)^2+n}{2x + \cfrac{(3+y)^2+n}{2x + \cfrac{(5+y)^2+n}{2x + \&c}}}$

then $\phi(x,y) = \phi(y,x)$.

~~Cor.~~

Sol. We get from $\overline{\underline{XIII}}$ 3 (a),

$$A_0 + \frac{x}{\underline{|1}}A_1 + \frac{x^2}{\underline{|2}}A_2 + \&c = e^x\left(A_0 - \frac{x}{\underline{|1}}A_1 + \frac{x^2}{\underline{|2}}A_2 - \&c\right)$$

$$\therefore (e^{x}+1)\left(A_0 + \frac{x}{\underline{|1}}A_1 + \frac{x^2}{\underline{|2}}A_2 + \&c\right) = (e^{x}+1)\left(A_0 - \frac{x}{\underline{|1}}A_1 + \frac{x^2}{\underline{|2}}A_2 - \&c\right)$$

$\therefore \left(A_0 + \frac{x}{\underline{|1}}A_1 + \frac{x^2}{\underline{|2}}A_2 + \&c\right)$ is an even function of x

Hence if n is even the coeff^t. of x^{n-1} must be 0.

$$\frac{1}{x^n} + \frac{n}{\underline{|1}}\cdot\frac{m}{n}\cdot\frac{1}{x^{n+1}} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{m(m+1)}{n(n+1)}\cdot\frac{1}{x^{n+2}} + \&c$$

$$= \frac{1}{(x-1)^n} + \frac{n}{\underline{|1}}\cdot\frac{m-n}{n}\cdot\frac{1}{(x-1)^{n+1}} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{(m-n)(m-n-1)}{n(n+1)}\cdot\frac{1}{(x-1)^{n+2}} + \&c$$

Sol. We have by

$$\frac{\underline{|n-1}\,\underline{|m+k-1}}{\underline{|n+k-1}\,\underline{|m-1}} = 1 + \frac{k}{\underline{|1}}\cdot\frac{m-n}{n} + \frac{k(k-1)}{\underline{|2}}\cdot\frac{(m-n)(m-n-1)}{n(n+1)} + \&c$$

multiplying both sides by $\frac{\underline{|n+k-1}}{\underline{|n-1}\,\underline{|k}}$ we have

$$\frac{\underline{|n+k-1}}{\underline{|k}\,\underline{|n-1}}\cdot\frac{\underline{|n-1}\,\underline{|m+k-1}}{\underline{|n+k-1}\,\underline{|m-1}} = \frac{\underline{|n+k-1}}{\underline{|n-1}\,\underline{|k}} + \frac{n}{\underline{|1}}\cdot\frac{m-n}{n}\cdot\frac{\underline{|n+k-1}}{\underline{|n}\,\underline{|k-1}} + \&c$$

The series in which L.H.S. is the coeff^t. of $\frac{1}{x^{n+k}}$

= that in which R.H.S. is the coeff^t. of $\frac{1}{x^{n+k}}$

$$\therefore e^x = \frac{1 + \frac{m}{n}\cdot\frac{x}{\underline{|1}} + \frac{m(m+1)}{n(n+1)}\cdot\frac{x^2}{\underline{|2}} + \frac{m(m+1)(m+2)}{n(n+1)(n+2)}\cdot\frac{x^3}{\underline{|3}} + \&c}{1 + \frac{m-n}{n}\cdot\frac{x}{\underline{|1}} + \frac{(m-n)(m-n-1)}{n(n+1)}\cdot\frac{x^2}{\underline{|2}} + \frac{(m-n)(m-n-1)(m-n-2)}{n(n+1)(n+2)}\cdot\frac{x^3}{\underline{|3}} + \&c}$$

Sol. multiply both sides in $\overline{\underline{XIII}}$ 4 by x^n & make x to ∞ & it can be shown that $A_n = 1$

$$\frac{1}{x^n} + \frac{n}{\underline{|1}}\cdot\frac{m}{2m}\cdot\frac{1}{x^{n+1}} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{m(m+1)}{2m(2m+1)}\cdot\frac{1}{x^{n+2}} + \&c$$

$$= \frac{1}{(x-1)^n} + \frac{n}{\underline{|1}}\cdot\frac{m}{2m}\cdot\frac{1}{(x-1)^{n+1}} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{m(m+1)}{2m(2m+1)}\cdot\frac{1}{(x-1)^{n+2}} + \&c$$

$$\therefore e^x = \frac{1 + \frac{m}{2m}\cdot\frac{x}{\underline{|1}} + \frac{m(m+1)}{2m(2m+1)}\cdot\frac{x^2}{\underline{|2}} + \frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}\cdot\frac{x^3}{\underline{|3}} + \&c}{1 - \frac{m}{2m}\cdot\frac{x}{\underline{|1}} + \frac{m(m+1)}{2m(2m+1)}\cdot\frac{x^2}{\underline{|2}} - \&c}$$

Ex 1 [illegible]

If
$$K=\frac{\left\lfloor\frac{x+l+m+n-1}{2}\right.\left\lfloor\frac{x+l-m-n-1}{2}\right.\left\lfloor\frac{x+m-n-l-1}{2}\right.\left\lfloor\frac{x+n-l-m-1}{2}\right.}{\left\lfloor\frac{x-l-m-n-1}{2}\right.\left\lfloor\frac{x-l+m+n-1}{2}\right.\left\lfloor\frac{x-m+n+l-1}{2}\right.\left\lfloor\frac{x-n+l+m-1}{2}\right.}$$

then
$$\frac{1-K}{1+K}=\frac{2lmn}{x^2-l^2-m^2-n^2+1+}\;\frac{4(l^2-1^2)(m^2-1^2)(n^2-1^2)}{3(x^2-l^2-m^2-n^2+5)+\&c}$$
$$+\frac{4(l^2-2^2)(m^2-2^2)(n^2-2^2)}{5(x^2-l^2-m^2-n^2+9)+\&c}=V$$

If
$$K=\frac{\left\lfloor\frac{x+l+n-1}{4}\right.\left\lfloor\frac{x+l-n-3}{4}\right.\left\lfloor\frac{x-l+n-3}{4}\right.\left\lfloor\frac{x-l-n-1}{4}\right.}{\left\lfloor\frac{x-l+n-1}{4}\right.\left\lfloor\frac{x-l-n-3}{4}\right.\left\lfloor\frac{x+l-n-1}{4}\right.\left\lfloor\frac{x+l+n-3}{4}\right.}$$

then
$$\frac{1-K}{1+K}=\cfrac{ln}{x^2-1-l^2+\cfrac{2^2-n^2}{1+\cfrac{2^2-l^2}{x^2-1+\cfrac{4^2-n^2}{1+\cfrac{4^2-l^2}{x^2-1+\&c}}}}}$$

If $\phi(y)=\frac{1}{y+1}-\frac{1}{y+3}+\frac{1}{y+5}+\&c$

then
$$\phi(x-l-n)-\phi(x+l-n)+\phi(x+l+n)-\phi(x-l+n)=\cfrac{2ln}{(x^2-1)+n^2-l^2+\cfrac{2(1^2-n^2)}{1+\cfrac{2(1^2-l^2)}{3(x^2-1)+n^2-l^2+\cfrac{4(2^2-n^2)}{1+\cfrac{4(2^2-l^2)}{\&c\ \&c}}}}}$$

2. $\sqrt{1-\frac{1}{x}} = \dfrac{1-(\frac{1}{2})^2\frac{1}{x-1}+(\frac{1.3}{2.4})^2\frac{1}{(x-1)^2}-(\frac{1.3.5}{2.4.6})^2\frac{1}{(x-1)^3}+\&c}{1+(\frac{1}{2})^2\frac{1}{x}+(\frac{1.3}{2.4})^2\frac{1}{x^2}+(\frac{1.3.5}{2.4.6})^2\frac{1}{x^3}+\&c}$

3. $1-(\frac{1}{2})^3+(\frac{1.3}{2.4})^3-(\frac{1.3.5}{2.4.6})^3+(\frac{1.3.5.7}{2.4.6.8})^3+\&c$

$=\{1-(\frac{1}{2})^2+(\frac{1.3}{2.4})^2-(\frac{1.3.5}{2.4.6})^2+\&c\}$

$\times\{1+(\frac{1}{2})^2\frac{1}{2}+(\frac{1.3}{2.4})^2\frac{1}{2^2}+(\frac{1.3.5}{2.4.6})^2\frac{1}{2^3}+\&c\}^2$

8. $\frac{1}{nx^n}+\frac{m}{\lfloor 1}\cdot\frac{1}{(n+1)x^{n+1}}+\frac{m(m+1)}{\lfloor 2}\cdot\frac{1}{(n+2)x^{n+2}}+\&c$

$=\frac{1}{n(x-1)^n}+\frac{m-n-1}{\lfloor 1}\cdot\frac{1}{(n+1)(x-1)^{n+1}}+\frac{(m-n-1)(m-n-2)}{\lfloor 2}\cdot\frac{1}{(n+2)(x-1)^{n+2}}+\&c$

9. $\frac{1}{nx^n}+\frac{n}{n(n+1)}\cdot\frac{1}{x^{n+1}}+\frac{n(n+1)}{n(n+1)(n+2)}\cdot\frac{1}{x^{n+2}}+\&c$

$=\frac{1}{n(x-1)^n}-\frac{n}{\lfloor 1}\cdot\frac{1}{(n+1)(x-1)^{n+1}}+\frac{n(n+1)}{\lfloor 2}\cdot\frac{1}{(n+2)(x-1)^{n+2}}-\&c$

10. $\frac{1}{nx}+\frac{\lfloor 1}{n(n+1)x^2}+\frac{\lfloor 2}{n(n+1)(n+2)x^3}+\&c$

$=\frac{1}{n(x-1)}-\frac{1}{(n+1)(x-1)^2}+\frac{\lfloor 2}{(n+2)(x-1)^3}-\&c$

11. $(1-x)^{a+b}\{1+\frac{ab}{1.n}x+\frac{a(a+1)b(b+1)}{1.2.n(n+1)}x^2+\&c\}$

$=(1-x)^n\{1+\frac{(n-a)(n-b)}{1.n}x+\frac{(n-a)(n-a+1)(n-b)(n-b+1)}{1.2.n(n+1)}x^2+\&c\}$

Sol. Apply XIII 4 twice.

12. $\frac{\lfloor x+y+n}{\lfloor x+n\,\lfloor y+n}+\frac{PA}{\lfloor 1}\cdot\frac{\lfloor x+y+n+1}{\lfloor x+n+1\,\lfloor y+n+1}+\frac{PA(P+1)(A+1)}{\lfloor 2}\cdot\frac{\lfloor x+y+n+2}{\lfloor x+n+2\,\lfloor y+n+2}$

$=\frac{\lfloor P+Q+n}{\lfloor P+n\,\lfloor Q+n}+\frac{xy}{\lfloor 1}\cdot\frac{\lfloor P+Q+n+1}{\lfloor P+n+1\,\lfloor Q+n+1}+\frac{x(x-1)y(y-1)}{\lfloor 2}\cdot\frac{\lfloor P+Q+n+2}{\lfloor P+n+2\,\lfloor Q+n+2}+\&c$

Sol. By XIII we have

$\frac{\lfloor x+y+n}{\lfloor x+n\,\lfloor y+n}$ ... [illegible]

$\frac{PA}{\lfloor 1}\,\frac{\lfloor x+y+n+1}{\lfloor x+n+1\,\lfloor y+n+1}=\frac{PA}{\lfloor 1}$ ... $+\frac{PA}{\lfloor 1}$ ... [illegible]

$$1 - \frac{n}{\underline{|1}}\cdot\frac{1}{2}\cdot\frac{1}{3} + \frac{n(n-1)}{\underline{|2}}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} - \&c$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\underline{|n}}{\underline{|n+\frac{1}{2}}}\left\{1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \&c \text{ to } n+1 \text{ terms}\right\}$$

$$\frac{\pi}{2}\left\{\frac{1}{n} + \left(\frac{1}{2}\right)^2\frac{1}{n+1} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{1}{n+2} + \&c\right\}$$

$$= \frac{1}{n} + \frac{n+\frac{1}{2}}{n(n+1)}\cdot\frac{1}{3} + \frac{(n+\frac{1}{2})(n+1\frac{1}{2})}{n(n+1)(n+2)}\cdot\frac{1}{5} + \&c$$

$$\frac{1}{n} + \frac{x}{\underline{|1}}\cdot\frac{y}{z}\cdot\frac{1}{n+1} + \frac{x(x-1)}{\underline{|2}}\cdot\frac{y(y-1)}{z(z+1)}\cdot\frac{1}{n+2} + \&c$$

$$= \frac{\underline{|z}\ \underline{|n-1}}{\underline{|z+n}}\left\{1 + \frac{n}{\underline{|1}}\cdot\frac{y+z}{z} + \frac{n(n+1)}{\underline{|2}}\cdot\frac{(y+z)(y+z+1)}{z(z+1)} + \&c \text{ to } x+1 \text{ terms}\right\}$$

$$\frac{p(p-1)}{2}\frac{q(q-1)}{2}\frac{\lfloor x+y+n+2}{\lfloor x+n+2\,\lfloor y+n+2} = \frac{p(p-1)q(q-1)}{\lfloor 2}\frac{1}{\lfloor n+2} + \&c$$

&c &c &c &c &c

Add up all the results

13. $$\frac{1}{y+n} + \frac{x}{1}\cdot\frac{y}{n}\cdot\frac{1}{p+n+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{n(n+1)}\cdot\frac{1}{p+n+2} + \&c$$

$$= \frac{\lfloor n-1\,\lfloor x+y+n}{\lfloor x+n\,\lfloor y+n} - p\,\frac{\lfloor n-1\,\lfloor x+y+n+1}{\lfloor x+n+1\,\lfloor y+n+1} + p(p-1)\frac{\lfloor n-1\,\lfloor x+y+n+2}{\lfloor x+n+2\,\lfloor y+n+2} - \&c$$

14. $$\frac{\pi}{4}\left\{\frac{1}{n+1} + \left(\frac{1}{2}\right)^2\frac{1}{n+2} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{n+3} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{n+4} + \&c\right\}$$

$$= 1 + \frac{n}{1}\left(\frac{2}{3}\right)^2 + \frac{n(n-1)}{\lfloor 2}\left(\frac{2.4}{3.5}\right)^2 + \frac{n(n-1)(n-2)}{\lfloor 3}\left(\frac{2.4.6}{3.5.7}\right)^2 + \&c$$

$$= \left(\frac{2.4.6\ldots\ldots 2n}{3.5.7\ldots\ldots(2n+1)}\right)^2\left\{1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c \text{ to } n+1 \text{ terms}\right\}$$

Cor. $$\frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\frac{1}{3} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{5} + \left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{7} + \&c\right\}$$

$$= \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c$$

2. $$\frac{\pi}{2}\left\{\frac{1}{n} + \left(\frac{1}{2}\right)^2\frac{1}{n+1} + \left(\frac{1.3}{2.4}\right)^2\frac{1}{n+2} + \&c\right\} - \log n$$

is finite when $n = \infty$

15. $$\frac{1}{y+n} - \frac{x}{n}\frac{1}{y+n+1} + \frac{x(x-1)}{n(n+1)}\frac{1}{y+n+2} - \&c$$

$$= \frac{1}{x+n} - \frac{y}{n}\frac{1}{x+n+1} + \frac{y(y-1)}{n(n+1)}\frac{1}{x+n+2} - \&c$$

16. $$\frac{1}{\{\phi(x)\}^2}\left[1 + \frac{n}{1}\frac{m}{2m}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\} + \frac{n(n-1)}{\lfloor 2}\frac{m(m+1)}{2m(2m+1)}\left\{\frac{\phi(x)-\phi(-x)}{\phi(x)}\right\}^2 + \&c\right]$$

is an even function of x

17. $$1 + \frac{n}{1}\frac{m}{2m}\left(\frac{2x}{1+x}\right) + \frac{n(n+1)}{\lfloor 2}\frac{m(m+1)}{2m(2m+1)}\left(\frac{2x}{1+x}\right)^2 + \&c$$

$$= (1+x)^n\left\{1 + \frac{n(n+1)}{2(2m+1)}x^2 + \frac{n(n+1)(n+2)(n+3)}{2.4.(2m+1)(2m+3)}x^4 + \&c\right\}$$

17. $$1 + \frac{n}{1}\frac{m}{2m}\frac{4x}{(1+x)^2} + \frac{n(n+1)}{\lfloor 2}\frac{m(m+1)}{2m(2m+1)}\left\{\frac{4x}{(1+x)^2}\right\}^2 + \&c$$

$$\phi(0) + \frac{2\phi'(0)}{\underline{|1}}\cdot\frac{m}{2m} + \frac{2^2\phi''(0)}{\underline{|2}}\cdot\frac{m(m+1)}{2m(2m+1)} + \&c$$

$$= \phi(1) + \frac{\phi''(1)}{2(2m+1)} + \frac{\phi^{iv}(1)}{2.4.(2m+1)(2m+3)} + \&c$$

If $m(m-1) = 2p$

$$e^{-mx}\left\{1 + \frac{1}{2}\cdot\frac{m}{\underline{|1}}(1-e^{-2x}) + \frac{1.3}{2.4}\cdot\frac{m(m+1)}{\underline{|2}}(1-e^{-2x})^2 + \&c\right\}$$

$$= 1 + \frac{A_1 x^2}{2(\underline{|1})^2} + \frac{A_2 x^4}{2^2(\underline{|2})^2} + \frac{A_3 x^6}{2^3(\underline{|3})^2} + \&c$$

where $A_n = p^n - \frac{n(n-1)}{\underline{|3}}p^{n-1} + \frac{n(n-1)(n-2)(3n-1)}{\underline{|5}}p^{n-2}$

$- \ldots \&c + (-1)^{n-1}2p\cdot\frac{\underline{|n-1}\,(2^{2n}-1)B_{2n}}{1.3.5\ldots(2n-1)} = \phi(p)$

$$\phi(1) = \frac{\underline{|n}}{1.3.5\ldots(2n-1)}$$

$$\left\{1 + \left(\tfrac{1}{2}\right)^2 x + \ldots\right\}^2 = \cfrac{1}{1-\cfrac{x}{2-\cfrac{3x}{8-\cfrac{5x}{2-\cfrac{17x}{40-\cfrac{23x}{2-\cfrac{1395x}{3128-\&c}}}}}}}$$

$= (1+[illegible])^{[illegible]}\left\{1 + \frac{2}{\lfloor 1}\cdot\frac{n-m+\frac{1}{2}}{m+\frac{1}{2}}\,x^2 + \frac{n(n+1)}{\lfloor 2}\cdot\frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}\,x^4 + \&c\right\}$

19. $1 + \frac{2x}{\lfloor 1}\cdot\frac{m}{2m} + \frac{(2x)^2}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)} + \frac{(2x)^3}{\lfloor 3}\cdot\frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)} + \&c$

$= e^{x}\left\{1 + \frac{x^2}{2}\cdot\frac{1}{2m+1} + \frac{x^4}{2\cdot 4}\cdot\frac{1}{(2m+1)(2m+3)} + \&c\right\}$

$[illegible]\; 1 + \frac{1}{2}\cdot\frac{x}{\lfloor 1} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^2}{\lfloor 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^3}{\lfloor 3} + \&c$

$= e^{\frac{x}{2}}\left\{1 + \frac{x^2}{4^2} + \frac{x^4}{4^2\cdot 8^2} + \frac{x^6}{4^2\cdot 8^2\cdot 12^2} + \&c\right\}$

Ex. 1. $1 - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi^2}{\lfloor 2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi^4}{\lfloor 4} - \&c = 0$

2. $1 - \frac{1\cdot 3}{1^2\cdot [illegible]}\,\pi^2 + \frac{1\cdot 3\cdot 5\cdot 7}{[illegible]}\,\pi^4 - \&c$

$= -\left(1 - \frac{\pi^2}{2^2} + \frac{\pi^4}{2^2\cdot 6^2} - \frac{\pi^6}{2^2\cdot 6^2\cdot 6^2} + \&c\right)$

3. $1 - \frac{1\cdot 3}{3\cdot 6}\cdot\frac{\pi^2}{\lfloor 2} + \frac{1\cdot 3\cdot 5\cdot 7}{3\cdot 6\cdot 9\cdot 12}\cdot\frac{\pi^4}{\lfloor 4} - \&c$

$= \frac{1}{2}\left(1 - \frac{\pi^2}{6^2} + \frac{\pi^4}{6^2\cdot 12^2} - \frac{\pi^6}{6^2\cdot 12^2\cdot 18^2} + \&c\right)$

20. i. $1 + \left(\frac{1}{2}\right)^2\frac{4x}{(1+x)^2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left\{\frac{4x}{(1+x)^2}\right\}^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2[illegible] + \&c$

$= (1+x)\left\{1 + \left(\frac{1}{2}\right)^2 x^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 x^6 + \&c\right\}$

ii. $1 + \left(\frac{1}{2}\right)^2\frac{2x}{1+x} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{2x}{1+x}\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\frac{2x}{1+x}\right)^3 + \&c$

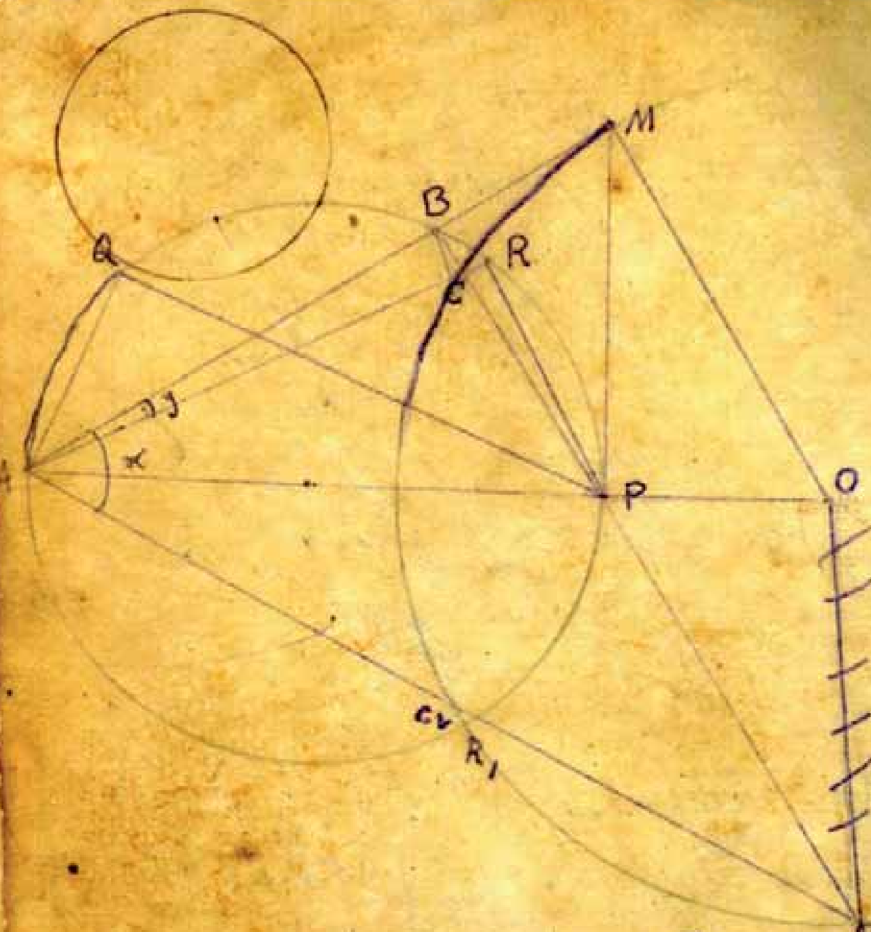
$= \sqrt{1+x}\left\{1 + \frac{1\cdot 3}{4^2}x^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}x^4 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4^2\cdot 8^2\cdot 12^2}x^6 + \&c\right\}$

E. $1 + \frac{2}{\lfloor 1}\cdot\frac{m}{2m}\cdot\frac{4x}{(1+x)^2} + \frac{n(n+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\left\{\frac{4x}{(1+x)^2}\right\}^2 + \&c$

$= \frac{1}{(1-x)^{[illegible]}}\left\{1 + \frac{[illegible]}{\lfloor 1}\cdot\frac{n-m+\frac{1}{2}}{[illegible]}\,x^2 + \frac{(2n-[illegible])[illegible]}{\lfloor 2}\right.$

$\left.\cdot\frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}\,x^4 + \&c\right\}$

[illegible] R. H. S. of XIII [illegible]



$$\frac{\dfrac{\beta}{\gamma}x + \dfrac{\alpha}{\lfloor 1}\cdot\dfrac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \dfrac{\alpha(\alpha-1)}{\lfloor 2}\cdot\dfrac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \&c}{1 + \dfrac{\alpha}{\lfloor 1}\cdot\dfrac{\beta}{\gamma}x + \dfrac{\alpha(\alpha+1)}{\lfloor 2}\cdot\dfrac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \&c}$$

$$= \frac{\beta x}{\gamma - (\alpha+\beta+1)x +}\;\frac{(\beta+1)(\alpha+\gamma+1)x}{\gamma+1-(\alpha+\beta+2)x +}\;\frac{(\beta+2)(\alpha+\gamma+2)x}{\gamma+2-(\alpha+\beta+3)x+\&c}$$

2. $1 + \frac{m}{2m}\cdot\frac{2x}{1+x} + \frac{n(n+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\left(\frac{2x}{1+x}\right)^2 + \&c$

$= \frac{(1+x)^m}{(1-x)^n}\left\{1 + \frac{(2m-n)(2m-n+1)}{2\cdot(2m+1)}x^2 + \frac{(2m-n)(2m-n+1)}{2\cdot 4}x\right.$

$\left.\frac{(2m-n+2)(2m-n+3)}{(2m+1)(2m+3)}x^4 + \&c\right\}$

Sol. Apply XIII 11 in R.H.S of XIII 17

3. $1 + \frac{n(n+1)}{2(2m+1)}\cdot\frac{4x}{(1+x)^2} + \frac{n(n+1)(n+2)(n+3)}{2\cdot 4(2m+1)(2m+3)}\left\{\frac{4x}{(1+x)^2}\right\}^2 + \&c$

$= (1+x)^n\left\{1 + \frac{n}{\lfloor 1}\cdot\frac{n-m+\frac{1}{2}}{m+\frac{1}{2}}x + \frac{n(n+1)}{\lfloor 2}\cdot\frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^2 + \&c\right.$

Sol. Combine the results of XIII 17 & 18.

4. $1 + \frac{n(n+1)}{\lfloor 2}\cdot\frac{x}{(1+x)^2} + \frac{n(n+1)(n+2)(n+3)}{1^2\cdot 2^2}\cdot\frac{x^2}{(1+x)^4} + \&c$

$= (1+x)^n\left\{1 + \frac{n^2}{(\lfloor 1)^2}x + \frac{n^2(n+1)^2}{(\lfloor 2)^2}x^2 + \frac{n^2(n+1)^2(n+2)^2}{(\lfloor 3)^2}x^3 + \&c\right\}$

21. $\left\{1 + \frac{x}{\lfloor 1}\cdot\frac{1}{m} + \frac{x^2}{\lfloor 2}\cdot\frac{1}{m(m+1)} + \frac{x^3}{\lfloor 3}\cdot\frac{1}{m(m+1)(m+2)} + \&c\right\}$

$\times\left\{1 + \frac{x}{\lfloor 1}\cdot\frac{1}{n} + \frac{x^2}{\lfloor 2}\cdot\frac{1}{n(n+1)} + \frac{x^3}{\lfloor 3}\cdot\frac{1}{n(n+1)(n+2)} + \&c\right\}$

$= 1 + \frac{x}{\lfloor 1}\cdot\frac{m+n}{mn} + \frac{x^2}{\lfloor 2}\cdot\frac{(m+n+1)(m+n+2)}{m(m+1)n(n+1)} +$

$\frac{x^3}{\lfloor 3}\cdot\frac{(m+n+2)(m+n+3)(m+n+4)}{m(m+1)(m+2)n(n+1)(n+2)} + \&c$

Sol. [illegible] we have

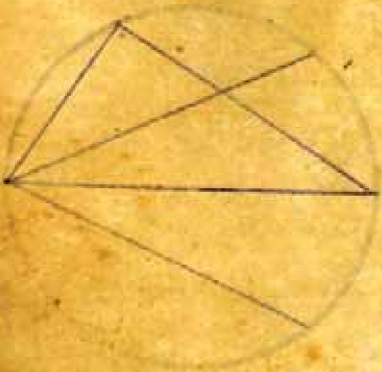
[illegible]

$= \frac{[illegible]\ x^r}{[illegible]}$

Equate [illegible] coefficients [illegible] ... [illegible]

[illegible] XIII [illegible]

$$\frac{1}{1+\dfrac{a_1 x}{1+\dfrac{a_2 x}{1+\dfrac{a_3 x}{1+ \&c}}}} = 1 - A_1 x + A_2 x^2 - A_3 x^3 + \&c$$



Let $P_n = a_1 a_2 \cdots a_{n-1}(a_1+a_2+\cdots a_n)$

$P_1 = A_1$

$P_2 = A_2$

$P_3 = A_3 - a_1 A_2$

$P_4 = A_4 - (a_1+a_2)A_3$

$P_5 = A_5 - (a_1+a_2+a_3)A_4 + a_1 a_3 A_3$

$P_6 = A_6 - (a_1+a_2+a_3+a_4)A_5 + (a_1a_3+a_1a_4+a_2a_4)A_4$

$P_n = \phi_0(n) A_n - \phi_1(n) A_{n-1} + \phi_2(n) A_{n-2} - \&c$

where $\phi_n(n+1) - \phi_n(n) = a_{n-1}\phi_{n-1}(n-1)$.

$$\frac{1}{1+b_1 x+\dfrac{a_1 x}{1+b_2 x+\dfrac{a_2 x}{1+b_3 x+\&c}}} = 1 - A_1 x + A_2 x^2 - A_3 x^3 + \&c$$

$P_n = a_1 a_2 a_3 \cdots a_{n-1}(\overline{a_1+b_1} + \overline{a_2+b_2} + \cdots + \overline{a_n+b_n})$

$= \phi_0(n) A_n - \phi_1(n) A_{n-1} + \phi_2(n) A_{n-2} - \&c$

where $\phi_n(n+1) - \phi_n(n) = b_n \phi_{n-1}(n) + a_{n-1}\phi_{n-1}(n-1)$.

$D_{n-1} = \phi_0(n) + x\phi_1(n) + x^2\phi_2(n) + x^3\phi_3(n) + \&c$.

Cor. 1. $e^{x}\left\{1+\frac{x^2}{2}\cdot\frac{1}{2n+1}+\frac{x^4}{2\cdot4}\cdot\frac{1}{(2n+1)(2n+3)}+\&c\right\}$

$$=1+\frac{2x}{\underline{1}}\cdot\frac{n}{2n}+\frac{(2x)^2}{\underline{2}}\cdot\frac{n(n+1)}{2n(2n+1)}+\&c$$

$$\left\{1+\frac{x}{4}\cdot\frac{1}{2n+1}+\frac{x^2}{4\cdot8}\cdot\frac{1}{(2n+1)(2n+3)}+\&c\right\}^2$$

$$=1+\frac{x}{\underline{1}}\cdot\frac{n}{2n}\cdot\frac{1}{2n+1}+\frac{x^2}{\underline{2}}\cdot\frac{n(n+1)}{2n(2n+1)}\cdot\frac{1}{(2n+1)(2n+3)}+\&c$$

Ex. 1. $1+\frac{1}{2}\cdot\frac{x}{\underline{1}}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^2}{\underline{2}}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^3}{\underline{3}}+\&c$

$$=e^{\frac{x}{2}}\left(1+\frac{x^2}{2^2}+\frac{x^4}{4^2\cdot[illegible]}+\frac{x^6}{[illegible]}+\&c\right)$$

2. $1+\frac{1}{2}\cdot\frac{x}{(\underline{1})^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^2}{(\underline{2})^2}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^3}{(\underline{3})^2}+\&c$

$$=\left(1+\frac{x}{2^2}+\frac{x^2}{2^2\cdot4^2}+\frac{x^3}{2^2\cdot4^2\cdot6^2}+\frac{x^4}{[illegible]}+\&c\right)^2$$

22. $\left\{1+\frac{x}{\underline{1}}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{2}}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}+\&c\right\}$

$$\times\left\{1-\frac{x}{\underline{1}}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{2}}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}-\&c\right\}$$

$$=1-\frac{x^2}{\underline{1}}\cdot\frac{m+n+3}{(m+1)(n+1)}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}$$

$$+\frac{x^4}{\underline{2}}\cdot\frac{(m+n+5)(m+n+6)}{(m+1)(m+2)(n+1)(n+2)}\cdot\frac{1}{(m+1)(m+2)(m+3)(m+4)}$$

$$\times\frac{1}{(n+1)(n+2)(n+3)(n+4)}-\frac{x^6}{\underline{3}}\cdot\frac{(m+n+7)(m+n+8)(m+n+9)}{(m+1)(m+2)(m+3)(n+1)(n+2)(n+3)}$$

$$\times\frac{1}{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)}\times\frac{1}{(n+1)(n+2)(n+3)(n+4)[illegible]}$$

$$+\&c$$

23. $\left\{1+\frac{x}{\underline{1}}\cdot\frac{1}{(m+n+1)}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{2}}\cdot\frac{1}{(m+n+1)(m+n+2)}\cdot\frac{1}{(n+1)(n+2)}+\&c\right\}$

$$\left\{1+\frac{x}{\underline{1}}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{2}}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}+\&c\right\}$$

$$=1+\frac{x}{\underline{1}}\cdot\frac{2m+n+2}{m+n+1}\cdot\frac{1}{m+1}\cdot\frac{n}{[illegible]}+\frac{x^2}{\underline{2}}\cdot\frac{[illegible]}{[illegible]}$$

$$\times\frac{1}{(m+1)[illegible]}+[illegible]$$

$$n\left\{1+\frac{x^n}{\lfloor n}+\frac{x^{2n}}{\lfloor 2n}+\frac{x^{3n}}{\lfloor 3n}+\&c\right\}$$

$$=e^x+e^{x\cos\frac{2\pi}{n}}\cos\left(x\sin\frac{2\pi}{n}\right)+e^{x\cos\frac{4\pi}{n}}\cos\left(x\sin\frac{4\pi}{n}\right)$$

$$+\ \&c \text{ to } n \text{ terms.}$$

$$\frac{x^h}{\lfloor h}+\left(\frac{x^{h+n}}{\lfloor h+n}+\frac{x^{h-n}}{\lfloor h-n}\right)+\left(\frac{x^{h+2n}}{\lfloor h+2n}+\frac{x^{h-2n}}{\lfloor h-2n}\right)+\&c$$

$$=1+\left(\frac{x^n}{\lfloor n}+\frac{x^{-n}}{\lfloor -n}\right)+\left(\frac{x^{2n}}{\lfloor 2n}+\frac{x^{-2n}}{\lfloor -2n}\right)+\&c$$

$$\left\{6m^2+(3m^3-m)\right\}^3+\left\{6m^2-(3m^3-m)\right\}^3$$

$$=\left\{6m^2(3m^2+1)\right\}^2.$$

$$\left\{m^7-3m^4(1+p)+m\left(3\,\overline{1+p}^{\,2}-1\right)\right\}^3$$

$$+\left\{2m^6-3m^3(1+2p)+(1+3p+3p^2)\right\}^3$$

$$+\left\{m^6-(1+3p+3p^2)\right\}^3$$

$$=\left\{m^7-3m^4p+m(3p^2-1)\right\}^3$$

$$\int_0^\infty\frac{\phi(x)}{\lfloor x}dx=\phi(0)+\frac{\phi(1)}{\lfloor 1}+\frac{\phi(2)}{\lfloor 2}+\frac{\phi(3)}{\lfloor 3}+\&c$$

$$\int_{-\infty}^{\infty}\frac{a^x}{\lfloor x}dx=e^a\ \ \int_{-\infty}^{\infty}\frac{\lfloor n}{\lfloor x\ \lfloor n-x}a^x dx=(1+a)^n.$$

$$+\frac{x^3}{\lfloor 3}\cdot\frac{(2m+n+5)(2m+n+7)(2m+n+9)}{(m+n+1)(m+n+2)(m+n+3)}\cdot\frac{1}{(m+1)(m+2)(m+3)}\times$$

$$\frac{1}{(n^2-1^2)(n^2-3^2)}+\frac{x^4}{\lfloor 4}\cdot\frac{(2m+n+6)(2m+n+8)(2m+n+10)(2m+n+12)}{(m+n+1)(m+n+2)(m+n+3)(m+n+4)}$$

$$\times\frac{1}{(m+1)(m+2)(m+3)(m+4)}\cdot\frac{1}{(n^2-2^2)(n^2-4^2)}+\&c$$

24. $\left\{1+\frac{x}{\lfloor 1}\cdot\frac{m}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{m(m+1)}{(n+1)(n+2)}+\&c\right\}$

$\times\left\{1-\frac{x}{\lfloor 1}\cdot\frac{m}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{m(m+1)}{(n+1)(n+2)}-\&c\right\}$

$=1-\frac{x^2}{\lfloor 2}\cdot\frac{m+n+1}{(n+1)(n+2)}\cdot\frac{2m}{n+1}+\frac{x^4}{\lfloor 4}\cdot\frac{(2m+n+1)(m+n+2)}{(n+1)(n+2)(n+3)(n+4)}$

$\times\frac{m(m-1)}{(n+1)(n+2)}+\&c$

25. $\left\{1+\frac{x}{\lfloor 1}mn+\frac{x^2}{\lfloor 2}m(m-1)n(n-1)+\&c\right\}$

$\times\left\{1-\frac{x}{\lfloor 1}mn+\frac{x^2}{\lfloor 2}m(m-1)n(n-1)-\&c\right\}$

$=1-\frac{x^2}{\lfloor 2}mn(m+n-1)+\frac{x^4}{\lfloor 4}m(m-1)n(n-1)(m+n-2)(m+n-3)$

$-\frac{x^6}{\lfloor 6}m(m-1)(m-2)n(n-1)(n-2)(m+n-3)(m+n-4)(m+n-5)+\&c$

26. $\left\{1+\frac{x}{\lfloor 1}\cdot\frac{m}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{m(m+1)}{(n+1)(n+2)}+\&c\right\}$

$\times\left\{1+\frac{x}{\lfloor 1}\cdot\frac{m+n}{n-1}+\frac{x^2}{\lfloor 2}\cdot\frac{(m+n)(m+n-1)}{(n-1)(n-2)}+\&c\right\}$

$=1+\frac{x}{\lfloor 1}(2m+n+1)\frac{n}{n^2-1^2}+\frac{x^2}{\lfloor 2}(2m+n)(2m+n+2)\frac{1}{n^2-2^2}$

$+\frac{x^3}{\lfloor 3}(2m+n-1)(2m+n+1)(2m+n+3)\frac{n^2}{(n^2-1^2)(n^2-3^2)}$

Ex. 1. $\left(1+\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}+\frac{x^9}{\lfloor 9}+\&c\right)\left(1-\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}-\frac{x^9}{\lfloor 9}+\&c\right)$

$=1+\dots\left\{1-\frac{(3x^2)^3}{\lfloor 6}+\frac{(3x^2)^6}{\lfloor 12}-\frac{(3x^2)^9}{\lfloor 18}+\&c\right\}$

2. $\left\{1+\frac{x}{(\lfloor 1)^3}+\frac{x^2}{(\lfloor 2)^3}+\frac{x^3}{(\lfloor 3)^3}+\&c\right\}\left\{1-\frac{x}{(\lfloor 1)^3}+\frac{x^2}{(\lfloor 2)^3}-\&c\right\}$

$=1-\frac{\lfloor 2}{(\lfloor 1\,\lfloor 2)^3}x^2+\frac{\lfloor 6}{(\lfloor 2\,\lfloor 4)^3}x^4-\frac{\lfloor 9}{(\lfloor 3\,\lfloor 6)^3}x^6+\&c$

$$1+\frac{1+x}{\lfloor 1}\cdot\frac{mn}{m+n+1}+\frac{(1+x)^2}{\lfloor 2}\cdot\frac{m(m+1)\,n(n+1)}{(m+n+1)(m+n+2)}+\&c$$

$$=\sqrt{\pi}\,\frac{\lfloor\frac{m+n-1}{2}}{\lfloor\frac{m-1}{2}\lfloor\frac{n-1}{2}}\left\{1+\frac{x^2}{\lfloor 2}mn+\frac{x^4}{\lfloor 4}m(m+2)\times n(n+2)+\&c\right\}$$

$$+2\sqrt{\pi}\,\frac{\lfloor\frac{m+n-1}{2}}{\lfloor\frac{m}{2}-1\lfloor\frac{n}{2}-1}\left\{\frac{x}{\lfloor 1}+\frac{x^3}{\lfloor 3}(m+1)(n+1)+\frac{x^5}{\lfloor 5}\times(m+1)(m+3)(n+1)(n+3)+\&c\right\}$$

$I=\alpha$, $II=\beta$, $IV=\gamma$ then

$$\frac{\sqrt{3}\,\sqrt[6]{\beta(1-\beta)}}{\sqrt[3]{\alpha(1-\gamma)}-\sqrt[3]{\gamma(1-\alpha)}}=\frac{\left\{1+\frac{1.2}{3^2}\alpha+\&c\right\}\left\{1+\frac{1.2}{3^2}\gamma+\&c\right\}}{\left\{1+\frac{1.2}{3^2}\beta+\&c\right\}^2}.$$

3. $\left(x + \frac{x^4}{\underline{|4}} + \frac{x^7}{\underline{|7}} + \frac{x^{10}}{\underline{|10}} + \&c\right)\left(x - \frac{x^4}{\underline{|4}} + \frac{x^7}{\underline{|7}} - \frac{x^{10}}{\underline{|10}} + \&c\right)$

$= \frac{1}{2}\left\{\frac{(2x^2)}{\underline{|2}} - \frac{(2x^2)^4}{\underline{|8}} + \frac{(2x^2)^7}{\underline{|14}} - \frac{(2x^2)^{10}}{\underline{|20}} + \&c\right\}$

4. [illegible] $\cosh x = 1 - \frac{(2x^2)^2}{\underline{|4}} + \frac{(2x^2)^4}{\underline{|8}} - \frac{(2x^2)^6}{\underline{|12}} + \&c$

5. [illegible] $\sinh x = \frac{(2x^2)}{\underline{|2}} - \frac{(2x^2)^3}{\underline{|6}} + \frac{(2x^2)^5}{\underline{|10}}\ \&c$

6. $\left\{1 + \frac{x}{(\underline{|1})^2} + \frac{x^2}{(\underline{|2})^2} + \frac{x^3}{(\underline{|3})^2} + \&c\right\}\left\{1 - \frac{x}{(\underline{|1})^2} + \frac{x^2}{(\underline{|2})^2} - \frac{x^3}{(\underline{|3})^2} + \&c\right\}$

$= 1 - \frac{x^2}{\underline{|2}}\cdot\frac{1}{(\underline{|1})^2} + \frac{x^4}{\underline{|4}}\cdot\frac{1}{(\underline{|2})^2} - \frac{x^6}{\underline{|6}}\cdot\frac{1}{(\underline{|3})^2} + \&c$

7. $\left\{1 + \frac{1}{2}\cdot\frac{x}{\underline{|1}} + \frac{1.3}{2.4}\cdot\frac{x^2}{\underline{|2}} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\underline{|3}} + \&c\right\}\left\{1 - \frac{1}{2}\cdot\frac{x}{\underline{|1}} + \frac{1.3}{2.4}\cdot\frac{x^2}{\underline{|2}}\ \&c\right\}$

$= 1 + \frac{1}{2}\cdot\frac{x^2}{2^2} + \frac{1.3}{2.4}\cdot\frac{x^4}{2^2.4^2} + \frac{1.3.5}{2.4.6}\cdot\frac{x^6}{2^2.4^2.6^2} + \&c$

8. $\left\{\frac{1}{n} + \frac{x}{n(n+1)} + \frac{x^2}{n(n+1)(n+2)} + \frac{x^3}{n(n+1)(n+2)(n+3)} + \&c\right\}$

$\times\left\{\frac{1}{n} - \frac{x}{n(n+1)} + \frac{x^2}{n(n+1)(n+2)} - \frac{x^3}{n(n+1)(n+2)(n+3)} + \&c\right\}$

$= \frac{1}{n}\cdot\frac{1}{n} + \frac{x^2}{n(n+1)(n+2)}\cdot\frac{1}{n+1} + \frac{x^4}{n(n+1)(n+2)(n+3)(n+4)}\cdot$ [illegible]

9. $\left(1 + \frac{x}{1.3} + \frac{x^2}{1.3.5} + \frac{x^3}{1.3.5.7} + \&c\right)\left(1 - \frac{x}{1.3} + \frac{x^2}{1.3.5} - \frac{x^3}{1.3.5.7}\right.$ [illegible]

$= 1 + \frac{x^2}{1.3.5}\cdot\frac{2}{3} + \frac{x^4}{1.3.5.7.9}\cdot\frac{4}{5} + \&c$

[illegible] $\left\{1 + xn + x^2 n(n-1) + x^3 n(n-1)(n-2) + \&c\right\}$

$\times\left\{1 - xn + x^2 n(n-1) - x^3 n(n-1)(n-2) + \&c\right\}$

$= $ [illegible] $+ \frac{x^2}{[illegible]} n(n-1)(n-2) + \frac{x^4}{[illegible]} n(n-1)(n-2)(n-3)(n-4) + \&c$

17. $1 + \frac{x}{1}\cdot\frac{y}{x+y+1} + \frac{x(x+1)}{\underline{|2}}\cdot\frac{y(y+1)}{(x+y+1)(x+y+2)}$

$+ \frac{x(x+1)(x+2)}{\underline{|3}}\cdot\frac{y(y+1)(y+2)}{(x+y+1)(x+y+2)(x+y+3)} + \&c = \frac{\underline{|x+y}}{\underline{|x}\ \underline{|y}}$ [illegible]

[illegible] 1 [illegible] $\sqrt{\frac{2\pi}{x}} = 1 + \frac{1^2 - x^2}{4(x+1)} + \frac{(1^2 - x^2)(3^2 - x^2)}{4.8.(x+1)(x+3)}$

$+ \frac{(1^2 - x^2)(3^2 - x^2)(5^2 - x^2)}{4.8.12.(x+1)(x+3)(x+5)} + \&c$

$$\int_0^\infty \frac{x^{n-1}}{1+x}\left\{1-\frac{\alpha\beta}{(\alpha+\beta)\underline{|1}}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)\underline{|2}}x^2-\&c\right\}dx$$

$$=\frac{\underline{|\alpha-n}\,\underline{|\beta-n}\,\underline{|n-1}}{\underline{|\alpha+\beta-n-1}}\left\{\frac{1}{\alpha+\beta-n}+\frac{\alpha\beta}{(\alpha+\beta)\underline{|1}}\cdot\frac{1}{\alpha+\beta-n+1}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)\underline{|2}}\cdot\frac{1}{\alpha+\beta-n+2}+\&c\right\}$$

$$=\frac{\underline{|\alpha-n}\,\underline{|n-1}}{\underline{|\alpha-1}}\left\{\frac{1}{\alpha}+\frac{\alpha n}{(\alpha+\beta)\underline{|1}}\cdot\frac{1}{\alpha+1}+\frac{\alpha(\alpha+1)n(n+1)}{(\alpha+\beta)(\alpha+\beta+1)\underline{|2}}\times\frac{1}{\alpha+2}+\&c\right\}.$$

If $\alpha+\beta=1$, then

$$\frac{1+\frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\gamma+1}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\underline{|2}(\gamma+1)(\gamma+2)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c}{1+\frac{\alpha+\gamma}{4}\cdot\frac{\beta+\gamma}{\gamma+1}x+\frac{(\alpha+\gamma)(\alpha+\gamma+2)(\beta+\gamma)(\beta+\gamma+2)}{4\cdot 8\,(\gamma+1)(\gamma+2)}x^2+\&c}$$

$$=\left(\frac{1+\sqrt{1-x}}{2}\right)^{\gamma}.$$

from XIII 11 & 29 or 29 alone.

2. $\frac{\lfloor\frac{n-1}{2}}{\left(\lfloor\frac{n-2}{4}\right)^2}\sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1^2}{4(n+1)} + \frac{1^2\cdot 3^2}{4\cdot 8\cdot(n+1)(n+3)} + \&c$

3. $\frac{\lfloor\frac{n}{2}}{\lfloor\frac{n-1}{4}\,\lfloor\frac{n+1}{4}}\sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1\cdot 3}{16(n+1)} + \frac{1\cdot 3\cdot 5\cdot 7}{16\cdot 32(n+1)(n+3)} + \&c$

28. If $x+y+z=n$, then

$$\frac{1}{x} + \frac{x}{1}\cdot\frac{y}{z}\cdot\frac{1}{x+1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{z(z+1)}\cdot\frac{1}{x+2} + \&c$$

$$= \frac{\lfloor n-1\,\lfloor x+y+n}{\lfloor x+n\,\lfloor y+n}\left\{1+\frac{x}{1}\cdot\frac{y}{z}+\frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{z(z+1)}+\&c \text{ to } x+y+n+1 \text{ terms}\right.$$

29. If $x+y+z=\frac{1}{2}$, then

$$1+\frac{x}{1}\cdot\frac{y}{z}p + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{z(z+1)}p^2 + \frac{x(x-1)(x-2)}{\lfloor 3}\cdot\frac{y(y-1)(y-2)}{z(z+1)(z+2)}p^3 + \&c$$

$$=1+\frac{2x}{1}\cdot\frac{2y}{z}\left(\frac{1-\sqrt{1-p}}{2}\right)+\frac{2x(2x-1)}{\lfloor 2}\cdot\frac{2y(2y-1)}{z(z+1)}\left(\frac{1-\sqrt{1-p}}{2}\right)^2+\&c$$

Cor. $1+\frac{1^2+n}{4^2}x + \frac{(1^2+n)(5^2+n)}{4^2\cdot 8^2}x^2 + \frac{(1^2+n)(5^2+n)(9^2+n)}{4^2\cdot 8^2\cdot 12^2}x^3+\&c$

$$=1+\frac{1^2+n}{2^2}\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{(1^2+n)(3^2+n)}{2^2\cdot 4^2}\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$$

30. If $x+y+z=0$, then

$$\left\{1+\frac{x}{1}\cdot\frac{y}{z+\frac{1}{2}}p+\frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{(z+\frac{1}{2})(z+1\frac{1}{2})}p^2+\&c\right\}^2$$

$$=1+\frac{2x}{1}\cdot\frac{2y}{2z+\frac{1}{2}}\cdot\frac{z}{2z}p+\frac{2x(2x-1)}{\lfloor 2}\cdot\frac{2y(2y-1)}{(2z+\frac{1}{2})(z+1\frac{1}{2})}p^2+\&c$$

Cor 1. $\left\{1+\frac{1^2+n}{4^2}x+\frac{(1^2+n)(5^2+n)}{4^2\cdot 8^2}x^2+\&c\right\}^2$

$$=1+\frac{1}{2}\cdot\frac{1^2+n}{2^2}x+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{(1^2+n)(3^2+n)}{2^2\cdot 4^2}x^2+\&c$$

$$\therefore\left(1+\frac{x}{2^2}+\frac{x^2}{2^2\cdot 4^2}+\frac{x^3}{2^2\cdot 4^2\cdot 6^2}+\&c\right)^2$$

$$=1+\frac{1}{2}\cdot\frac{x}{(\lfloor 1)^2}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^2}{(\lfloor 2)^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^3}{(\lfloor 3)^2}+\&c$$

× If pth & qth be $\phi(x)$ & $\psi(x)$ and nth be $f(x)$
then (i) if pth & qth be $\phi F(x)$ & $\psi F(x)$ then nth $= f F(x)$.
(ii) if pth & qth be $F\phi(x)$ & $F\psi(x)$ then nth $= Ff(x)$.
Thus we may add or subtract any constant
(& multiply or divide by any constant)
to x in each function or to each function

$I = x$ $\quad II = x^2 + 2x$ then nth $= (x+1)^n - 1$

$I = x$ $\quad II = x^2 + 4x$ then nth $= \left\{\left(\frac{\sqrt{x+4}+\sqrt{x}}{2}\right)^n - \left(\frac{\sqrt{x+4}-\sqrt{x}}{2}\right)^n\right\}^2$

$I = x$ $\quad II = x^2 - 2$ then nth $= \left(\frac{x+\sqrt{x^2-4}}{2}\right)^n + \left(\frac{x-\sqrt{x^2-4}}{2}\right)^n$

$I = x$ $\quad II = \frac{x^2}{(2-x)^2}$ $\quad n$th $= \frac{4x^n}{\left\{(1+\sqrt{1-x})^n + (1-\sqrt{1-x})^n\right\}^2}$

If $I = x$ & II $x^2 + 2nx$, then

$$III = x^3 + 3nx^2 + 3 \cdot \frac{n(n+1)}{2} x - \frac{n(n-1)(n-2)x}{2x + \frac{3(n+1)}{2}}.$$

If $\chi(x)$ y be the common equation then the required series $= \frac{y}{x(1-x)\,y_x}$ or any power of this or more generally if $f(x)$ & $F(x)$ be of the pth & qth degree find $\phi(x)$ such that $\sqrt[p]{\phi f(x)} = \sqrt[q]{\phi F(x)} = \chi(x)$, then

CHAPTER XIV

1. $\frac{a_1}{b_1+\frac{a_2}{b_2+\frac{a_3}{b_3+\dots+\frac{a_n}{b_n}}}} = a_1\frac{N_{n-1}}{D_n} = \frac{a_1}{D_0D_1} - \frac{a_1a_2}{D_1D_2} + \frac{a_1a_2a_3}{D_2D_3} - \&c$ to n terms.

$N_{n-1} = b_n N_{n-2} + a_n N_{n-3}$ and $D_n = b_n D_{n-1} + a_n D_{n-2}$

Cor. $a_1+a_2+a_3+\&c$ to n terms $= \frac{a_1}{1-\frac{a_2}{a_1+a_2-\frac{a_1a_3}{a_2+a_3-\frac{a_2a_4}{a_3+a_4-\frac{a_3a_5}{a_4+a_5-\&c}}}}}$ to n terms

2. $x = x-a_1+\frac{xa_1}{x-a_2+\frac{xa_2}{x-a_3+\frac{xa_3}{x-a_4+\&c}}}$

3. $x = a_1+\sqrt{x^2-a_1(a_2+2x)-2a_1\sqrt{x^2+a_2(a_3+2x)-2a_2\sqrt{\&c}}}$

4. $x+n+a = \sqrt{ax+(n+a)^2+x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{\&c}}}$

Ex. 1. $3 = 1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\&c}}}}$

2. $4 = 1\sqrt{6+2\sqrt{7+3\sqrt{8+\&c}}}$

5. $\frac{1}{x+a} - \frac{1}{(x+a)(x+2a)} + \frac{1}{(x+a)(x+2a)(x+3a)} - \&c$

$= \frac{1}{x+a+\frac{x+a}{x+2a-1+\frac{x+2a}{x+3a-1+\frac{x+3a}{x+4a-1+\&c}}}}$

Cor. $\frac{1}{e-1} = \frac{1}{1+\frac{2}{2+\frac{3}{3+\&c}}}$

6. $x = \frac{x^2}{x+\frac{x+2}{2x+1+\frac{x+3}{3x+2+\&c}}}$

Cor. $1 = \frac{1}{1+\frac{3}{1+\frac{4}{5+\&c}}}$

7. [illegible]

$x+3$ [illegible]

the function for the n th degree $= \phi^{-1}\{\chi(x)\}^{n}$,

The self-repeating series $= \sqrt[n]{\dfrac{\phi(x)}{\psi(x)\,\phi'(x)}}$ where n is any quantity and $\psi(x)$ any known function.

Supposing the series to be ~~agree~~ $S(x)$. then

$$\frac{S\,F(x)}{S\,f(x)} = \sqrt[n]{\frac{p}{q}\cdot\frac{\psi f(x)\,F'(x)}{\psi F(x)\,f'(x)}}$$

If $y = e^{-2\pi\cdot\frac{1+\frac{1\cdot5}{6^2}(1-x)+\&c}{1+\frac{1\cdot5}{6^2}x+\&c}}$

then
$$1-504\left(\frac{1^5y}{1-y}+\frac{2^5y^2}{1-y^2}+\frac{3^5y^3}{1-y^3}+\&c\right)$$
$$=\left\{1+\frac{1\cdot5}{6^2}x+\&c\right\}^6(1-2x).$$

If $y = e^{-\frac{2\pi}{\sqrt{3}}\cdot\frac{1+\frac{1\cdot2}{3^2}(1-x)+\&c}{1+\frac{1\cdot2}{3^2}x+\&c}}$

then
$$1+240\left(\frac{1^3y}{1-y}+\frac{2^3y^2}{1-y^2}+\frac{3^3y^3}{1-y^3}+\&c\right)$$
$$=\left\{1+\frac{1\cdot2}{3^2}x+\&c\right\}^4(1+8x).$$
$$1-504\left(\frac{1^5y}{1-y}+\frac{2^5y^2}{1-y^2}+\frac{3^5y^3}{1-y^3}+\&c\right)$$
$$=\left\{1+\frac{1\cdot2}{3^2}x+\&c\right\}^6(1-20x-8x^2)$$

If $y = e^{-\pi\sqrt{2}\cdot\frac{1+\frac{1\cdot3}{4^2}(1-x)+\&c}{1+\frac{1\cdot3}{4^2}x+\&c}}$. then

$$1+240\left(\frac{1^3y}{1-y}+\frac{2^3y^2}{1-y^2}+\&c\right)=\left(1+\frac{1\cdot3}{4^2}x+\&c\right)^4(1+3x).$$
$$1-504\left(\frac{1^5y}{1-y}+\frac{2^5y^2}{1-y^2}+\&c\right)=\left(1+\frac{1\cdot3}{4^2}x+\&c\right)^6(1-9x).$$

Cor. 1. $\frac{4}{\pi} = \frac{3}{1+\frac{4}{2+\frac{5}{3+\frac{6}{4+\text{\&c}}}}}$

2. $\frac{4}{\pi} = \frac{4}{1+\frac{6}{3+\frac{8}{5+\frac{10}{7+\text{\&c}}}}}$

8. [illegible] is a positive integer,

$[illegible] = \frac{1}{1-x} + \frac{2}{2-x} + \frac{3}{3-x} + \cdots + \frac{x}{0} + \frac{x+1}{1} + \frac{x+2}{2} + \text{\&c}$

9. If a is a positive integer and if

$\frac{N_a}{D_a} = \frac{n}{n-a+\frac{n+1}{n-a+1+\frac{n+2}{n-a+2+\text{\&c}}}}$, then

$\frac{N_{a+1}}{N_a} = n+2-a + \frac{a-1}{n+3-a+\frac{a-2}{n+4-a+\text{\&c}}}$

Here we should equate the numerators and the denominators (in [illegible])

If $N = \phi(n)$, then $D = \phi(n-1)$.

Cor. 1. $\frac{n^2+n+1}{n^2-n+1} = \frac{n}{n-3+\frac{n+1}{n-2+\frac{n+2}{n-1+\text{\&c}}}}$

2. $\frac{n^3+2n+1}{(n-1)^3+2(n-1)+1} = \frac{n}{n-4+\frac{n+1}{n-3+\frac{n+2}{n-2+\text{\&c}}}}$

10. $1 = \frac{x+a}{a+\frac{(x+a)^2-a^2}{a+\frac{(x+2a)^2-a^2}{a+\text{\&c}}}}$

11. If [illegible] $= b$, then

$a = \frac{ab}{a+b+d-\frac{(a+d)(b+d)}{a+b+3d-\frac{[illegible]}{[illegible]}}}$

12. $\frac{a_1}{x+\frac{a_2}{1+\frac{a_3}{x+\frac{a_4}{1+\text{\&c to } n \text{ terms}}}}}$

$= [illegible]$

[illegible]

$$\sqrt{2}\left\{\frac{1}{2} + e^{-\frac{\pi x}{x^2+y^2}}\cos\left(\frac{\pi y}{x^2+y^2}\right) + e^{-\frac{4\pi x}{x^2+y^2}}\cos\left(\frac{4\pi y}{x^2+y^2}\right) + \&c\right\}$$

$$= \sqrt{\sqrt{x^2+y^2}+x}\left\{\frac{1}{2} + e^{-\pi x}\cos \pi y + e^{-4\pi x}\cos 4\pi y + \&c\right\}$$

$$+ \sqrt{\sqrt{x^2+y^2}-x}\left\{e^{-\pi x}\sin \pi y + e^{-4\pi x}\sin 4\pi y + \&c\right\}$$

$$\sqrt{2}\left\{e^{-\frac{\pi x}{x^2+y^2}}\sin\left(\frac{\pi y}{x^2+y^2}\right) + e^{-\frac{4\pi x}{x^2+y^2}}\sin\left(\frac{4\pi y}{x^2+y^2}\right) + \&c\right\}$$

$$= \sqrt{\sqrt{x^2+y^2}-x}\left\{\frac{1}{2} + e^{-\pi x}\cos \pi y + e^{-4\pi x}\cos 4\pi y + \&c\right\}$$

$$- \sqrt{\sqrt{x^2+y^2}+x}\left\{e^{-\pi x}\sin \pi y + e^{-4\pi x}\sin 4\pi y + \&c\right\}$$

$$\frac{1}{1^2} - \frac{4}{3^2} + \frac{4}{5^2} - \&c = .915\,965,\,594\,177$$

$$\alpha = \frac{27p(1+p)^4}{2(1+4p+p^2)^3} \cdot \alpha\beta = \frac{27p^4(1+p)}{2(2+2p-p^2)^3} \text{ then}$$

$$\left(1+p-\frac{p^2}{2}\right)\left\{1 + \frac{1.2}{3^2}\alpha + \frac{1.2.4.5}{3^2.6^2}\alpha^2 + \&c\right\}$$

$$= (1+4p+p^2)\left\{1 + \frac{1.2}{3^2}\beta + \frac{1.2.4.5}{3^2.6^2}\beta^2 + \&c\right\}$$

13. $\dfrac{a_1+h}{1+\dfrac{a_1}{x+\dfrac{a_2+h}{1+\dfrac{a_2}{x+\dfrac{a_3+h}{1+\&c}}}}}$
$= h+\dfrac{a_1}{1+\dfrac{a_1+h}{x+\dfrac{a_2}{1+\dfrac{a_2+h}{x+\&c}}}}$

14. $\dfrac{1}{(m+1)(n+1)}-\dfrac{1}{(m+2)(n+2)}+\&c$
$=\dfrac{1}{m+n+1+mn+\dfrac{(m+1)^2(n+1)^2}{m+n+3+\dfrac{(m+2)^2(n+2)^2}{m+n+5+\&c}}}$

15. $\dfrac{a_1x}{b_1+\dfrac{a_2x}{b_2+\dfrac{a_3x}{b_3+\&c}}} = T_1x-T_2x^2+T_3x^3-\&c$

Let $\dfrac{P_n}{b_n}=\dfrac{a_1a_2\cdots a_n}{(b_1b_2\cdots b_n)^2}$ and $T_n-P_n=t_n$, then

$t_1=0,\ t_2=0$

$T_1t_3-T_2^2=0$

$T_2t_4-T_3^2=0$

$T_3t_5-T_4^2=\dfrac{M^2}{P_1P_3}$ where $M=T_2T_4-T_3^2$

$T_4t_6-T_5^2=\dfrac{N^2}{P_2P_4}$ where $N=T_3T_5-T_4^2$

16. $\dfrac{(x+1)^n-(x-1)^n}{(x+1)^n+(x-1)^n}=\dfrac{n}{x+\dfrac{n^2-1^2}{3x+\dfrac{n^2-2^2}{5x+\dfrac{n^2-3^2}{7x+\&c}}}} = A_n.$ $\qquad A_n+\dfrac{1}{A_n}=\dfrac{2}{A_{2n}}$

Cor. $\tan^{-1}\dfrac{1}{x}=\dfrac{1}{x+\dfrac{1^2}{3x+\dfrac{2^2}{5x+\dfrac{3^2}{7x+\&c}}}}$

$\tfrac{1}{2}\log\dfrac{x+1}{x-1}=\dfrac{1}{x-\dfrac{1^2}{3x-\dfrac{2^2}{5x-\dfrac{3^2}{7x-\&c}}}}$

$\tan\dfrac{1}{x}=\dfrac{1}{x-\dfrac{1}{3x-\dfrac{1}{5x-\dfrac{1}{7x-\&c}}}}$

$\dfrac{e^{\frac{2}{x}}-1}{e^{\frac{2}{x}}+1}=\dfrac{1}{x+}$ [illegible]

$$e^{-\pi\sqrt{2}} \cdot \frac{1 + \frac{1.3}{4^2}(1-x) + }{1 + \frac{1.3}{4^2} x + } = y$$

III degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 4\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

VII degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 20\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$

$$+ 8\sqrt{2}\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\} = 1$$

$$\left\{1 + \frac{1.3}{4^2}x + \frac{1.3.5.7}{4^2.8^2}x^2 + \&c\right\}^2$$

$$= 1 + 24\left(\frac{y}{1-y} + \frac{3y^3}{1-y^3} + \frac{5y^5}{1-y^5} + \&c\right)$$

V degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 8\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right)$

$$= 1$$

$$F\left(\frac{3\sqrt{2} - \sqrt{5} - 2}{3\sqrt{2} + \sqrt{5} + 2}\right) = e^{-\pi\sqrt{10}} = F\left\{(\sqrt{10} - 3)^2(\sqrt{2} - 1)^4\right\}$$

$$F\left(\frac{7\sqrt{2} - 2\sqrt{6} - 5}{7\sqrt{2} + 2\sqrt{6} + 5}\right) = e^{-3\pi\sqrt{2}}$$

$$1 + \frac{1.2}{3^2}\left\{1 - \left(\frac{1-t}{1+2t}\right)^3\right\} + \&c$$

$$= (1 + 2t)\left\{1 + \frac{1.2}{3^2}t^3 + \frac{1.2.4.5}{3^2.6^2}t^6 + \&c\right\}$$

XI $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 16\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[3]{\alpha\beta} + \sqrt[3]{1-\alpha . 1-\beta}\right)$

$$+ 48\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right)$$

$$+ 68\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} = 1$$

7. $\dfrac{\frac{x}{n} + \frac{x^2}{\lfloor 2} \cdot \frac{1}{n(n+1)} + \frac{x^3}{\lfloor 3} \cdot \frac{1}{n(n+1)(n+2)} + \&c}{1 + \frac{x}{\lfloor 1} \cdot \frac{1}{n} + \frac{x^2}{\lfloor 2} \cdot \frac{1}{n(n+1)} + \&c}$

$= \dfrac{x}{n + \frac{x}{n+1 + \frac{x}{n+2 + \frac{x}{n+3 + \&c}}}}$

Sol. Let $\phi(x) = \frac{x}{\lfloor 0 \lfloor 1} + \frac{x^2}{\lfloor 1 \lfloor 2} + \frac{x^3}{\lfloor 2 \lfloor 3} + \frac{x^4}{\lfloor 3 \lfloor 4} + \&c$, then we see that

$\phi(x) = x\phi''(x)$

$\phi'(x) = \phi''(x) + x\phi'''(x)$

$\phi''(x) = 2\phi'''(x) + x\phi^{iv}(x)$

$\phi'''(x) = 3\phi^{iv}(x) + x\phi^{v}(x)$

&c &c

$\dfrac{\phi(x)}{\phi'(x)} = \dfrac{x\phi''(x)}{\phi''(x) + x\phi'''(x)} = \dfrac{x}{1 + \frac{x\phi'''(x)}{\phi''(x)}} = \dfrac{x}{1 + \frac{x\phi'''(x)}{2\phi'''(x) + x\phi^{iv}(x)}}$

$= \dfrac{x}{1 + \frac{x}{2 + x\frac{\phi^{iv}(x)}{\phi'''(x)}}}$

$= \dfrac{x}{1 + \frac{x}{2 + \frac{x}{3 + \cdots + \frac{x}{n-1 + \cdots}}}}$ ... $\dfrac{\frac{x}{n} + \frac{x^2}{\lfloor 2} \cdot \frac{1}{n(n+1)} + \&c}{1 + \frac{x}{n} \cdots}$

18. $\dfrac{\frac{m}{n}x + \frac{a-n}{\lfloor 2} \cdot \frac{m(m+1)}{n(n+1)} x^2 + \frac{(a-n)(a-n-1)}{\lfloor 3} \cdot \frac{m(m+1)(m+2)}{n(n+1)(n+2)} \cdots}{1 + \frac{a-n}{\lfloor 1} \cdot \frac{m}{n} x + \frac{(a-n)(a-n-1)}{\lfloor 2} \cdots}$

$= \dfrac{x\,m\,a}{n + \frac{x(m-n)(a-n)}{n+1 + \frac{x(m+1)(a+1)}{n+2 + \frac{x(m-n-1)(a-n-1)}{n+3 + \&c}}}}$

19. $\frac{m}{n}x + \frac{m(m+1)}{n(n+1)}x^2 + \&c$

$= \dfrac{m\,x}{n - \frac{m(n-1)x}{n+1 + \frac{(m-n)x}{n+2 \cdots}}}$

$+ \frac{\lfloor 2 (\ldots+2)}{\ldots} \cdot \frac{(n+1)x}{\ldots}$

$= \dfrac{n\,x}{m - \frac{x(n+1)}{\ldots} - \frac{(n+1)x(\ldots-2)}{n+1 + x(n+1) \cdots}}$

If $y = e^{-2\pi \cdot \frac{1 + \frac{1.5}{6^2}(1-x) + \frac{1.5.7.11}{6^2.12^2}(1-x)^2 + \&c}{1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c}}$

then $\left\{1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c\right\}^4$

$= 1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \frac{3^3 y^3}{1-y^3} + \&c\right)$

If $1 + \frac{1.5}{6^2}x + \frac{1.5.7.11}{6^2.12^2}x^2 + \&c$

$= \left\{1 + \left(\frac{1}{2}\right)^2 y + \left(\frac{1.3}{2.6}\right)^2 y^2 + \&c\right\}\sqrt[4]{1-y+y^2}$

& $x(1-x) = A$ & $y(1-y) = B$

then $A = \frac{27B^2}{16(1-B)^3}$

If $y = \frac{p(2+p)}{1+2p}$ then $x = \frac{27}{4} \cdot \frac{(p+p^2)^2}{(1+p+p^2)^3}$

If $z = \frac{p^3(2+p)}{1+2p}$ and $x = \frac{27}{4} \cdot \frac{(p+p^2)^2}{(1+p+p^2)^3}$

then $1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^2.6^2}x^2 + \&c$

$= \left\{1 + \left(\frac{1}{2}\right)^2 z + \left(\frac{1.3}{2.4}\right)^2 z^2 + \&c\right\}\frac{1+p+p^2}{\sqrt{1+2p}}$

Sol. Write $n+1$ for n in Ex. 13. Subtract both sides from 1 and equate the reciprocals of the results. 197

Cor. 1. $\frac{x}{m} + \frac{x^2}{m(m+1)} + \frac{x^3}{m(m+1)(m+2)} + \&c.$

$= \cfrac{x}{m - \cfrac{mx}{m+1+\cfrac{1x}{m+2-\cfrac{(m+1)x}{m+3+\cfrac{2x}{m+4-\&c.}}}}}$

$= \cfrac{x}{m-x+\cfrac{x}{m+1-x+\cfrac{2x}{m+2-x+\cfrac{3x}{m+3-x+\&c.}}}}$

$1 + \frac{x}{x+1} + \frac{x^2}{(x+1)(x+2)} + \&c.$

$= \cfrac{1}{1 - \cfrac{x}{1+\cfrac{2x}{1+\cfrac{3x}{2+\&c.}}}}$

20. $\dfrac{\left\lfloor \frac{x+m+n-1}{2} \right. \left\lfloor \frac{x-m-n-1}{2} \right. - \left\lfloor \frac{x+m-n-1}{2} \right. \left\lfloor \frac{x-m+n-1}{2} \right.}{\left\lfloor \frac{x+m+n-1}{2} \right. \left\lfloor \frac{x-m-n-1}{2} \right. + \left\lfloor \frac{x+m-n-1}{2} \right. \left\lfloor \frac{x-m+n-1}{2} \right.}$

$= \cfrac{mn}{x + \cfrac{(m^2-1^2)(n^2-1^2)}{3x + \cfrac{(m^2-2^2)(n^2-2^2)}{5x + \cfrac{(m^2-3^2)(n^2-3^2)}{7x + \&c.}}}}$

21. $\dfrac{\left\lfloor \frac{x+n-1}{2} \right. \left\lfloor \frac{x-n-1}{2} \right.}{\left\lfloor \frac{x+n-1}{2} \right. \left\lfloor \frac{x-n-1}{2} \right.} = \cfrac{4}{x - \cfrac{n^2-1^2}{2x - \cfrac{n^2-3^2}{2x - \cfrac{n^2-5^2}{2x}}}}$

22. $\left(\dfrac{\left\lfloor \frac{x-1}{2} \right.}{\left\lfloor \frac{x-2}{2} \right.}\right)^2 = \cfrac{4}{x + \cfrac{1^2}{2x + \cfrac{3^2}{2x + \cfrac{5^2}{2x + \&c.}}}}$

23. $\dfrac{\left\lfloor \frac{x-3}{4} \right. \left\lfloor \frac{x-1}{4} \right.}{\left\lfloor \frac{x-1}{4} \right. \left\lfloor \frac{x-3}{4} \right.} = \cfrac{8}{x + \cfrac{1^2}{2x + \&c.}}$

24. $\left\lfloor \frac{x+n-3}{4} \right. \left\lfloor \frac{x-n-1}{4} \right. \ldots$

$$\alpha = \frac{p(3+p)^2}{2(1+p)^3}; \quad \beta = \frac{p^2(3+p)}{4}$$

$$1-\alpha = \frac{(1-p)^2(2+p)}{2(1+p)^3}, \quad 1-\beta = \frac{(1-p)(2+p)^2}{4}.$$

$$1 + \frac{1.2}{3^2}\alpha + \frac{1.2.4.5}{3^2.6^2}\alpha^2 + \frac{1.2.4.5.7.8}{3^2.6^2.9^2}\alpha^3 + \&c$$

$$= (1+p)\left\{1 + \frac{1.2}{3^2}\beta + \frac{1.2.4.5}{3^2.6^2}\beta^2 + \frac{1.2.4.5.7.8}{3^2.6^2.9^2}\beta^3 + \&c\right\}$$

II deg. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} = 1$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} - \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{2}{1+p}$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} - \sqrt[3]{\frac{\beta^2}{\alpha}} = 1+p.$$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} + \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{4}{(1+p)^2}$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} + \sqrt[3]{\frac{\beta^2}{\alpha}} = (1+p)^2$$

IV deg. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

XI. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt{3}\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right\}$
$+ 6\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1.$

1, 2, 4, 8

$\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)} + 3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)} \times$

$$\frac{1 + \frac{1.2}{3^2}\beta + \&c}{1 + \frac{1.2}{3^2}\alpha + \&c}, \quad \frac{1 + \frac{1.2}{3^2}\gamma + }{1 + \frac{1.2}{3^2}\delta + } = 1$$

23. $\left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \frac{1}{x+n+7} + \&c\right)$

$- \left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \frac{1}{x-n+7} + \&c\right)$

$= [illegible] + \cfrac{1^2-n^2}{x + \cfrac{2^2}{x + \cfrac{3^2-n^2}{x + \cfrac{4^2}{x + \&c}}}}$

Cor. $2\left(\frac{1}{x+1} - \frac{1}{x+3} + \frac{1}{x+5} - \frac{1}{x+7} + \&c\right) = \cfrac{1}{x + \cfrac{1^2}{x + \cfrac{2^2}{x + \cfrac{3^2}{x + \&c}}}}$

24. $\left(\frac{1}{x-n+1} + \frac{1}{x-n+3} + \frac{1}{x-n+5} + \&c\right)$

$- \left(\frac{1}{x+n+1} + \frac{1}{x+n+3} + \frac{1}{x+n+5} + \&c\right)$

$= \cfrac{n}{x + \cfrac{1^2(1^2-n^2)}{3x + \cfrac{2^2(2^2-n^2)}{5x + \cfrac{3^2(3^2-n^2)}{7x + \&c}}}}$

Cor. $2\left\{\frac{1}{(x+1)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+5)^2} + \&c\right\}$

$= [illegible]$

25. $2x^2\left\{\frac{1}{2x^2} - \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} + \&c\right\}$

$= \cfrac{1}{x + \cfrac{1^2}{x + \cfrac{1\cdot 2}{x + \cfrac{2^2}{x + \cfrac{2\cdot 3}{x + \cfrac{3^2}{x + \&c}}}}}}$

Cor. $2x\left(\frac{1}{2x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \&c\right)$

$= [illegible]$

$\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + [illegible]$

$+ \frac{1^3}{[illegible]}$

$+ \frac{2^3}{[illegible]}$

$\frac{2^6}{[illegible]}$

If $y = e^{-\frac{2\pi}{\sqrt{3}} \cdot \frac{1 + \frac{1.2}{3^2}(1-x) + \frac{1.2.4.5}{3^2.6^2}(1-x)^2 + \&c}{1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^2.6^2}x^2 + \&c}}$

then $1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^2.6^2}x^2 + \&c$

$= 1 + 6\left(\frac{y}{1-y} - \frac{y^2}{1-y^2} + \frac{y^4}{1-y^4} - \frac{y^5}{1-y^5} + \&c\right)$

$1 + 12\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \&c\right)$

$= \left\{1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5} + \&c\right\}^2\right.$

$1 + 4\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{6x^6}{1-x^6} + \frac{8x^8}{1-x^8}\right.$

$= \left\{1 + 2\left(\frac{x}{1-x} + \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5} - \frac{x^6}{1-x^6}\right.\right.$

$\left.\left. + \frac{x^8}{1-x^8} + \frac{x^9}{1-x^9} - \frac{x^{10}}{1-x^{10}} + \frac{x^{11}}{1-x^{11}} - \&c\right)\right\}^2$

$1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \&c\right)$

$= 4\frac{\psi^3(x^2)}{\psi(x^6)} - 3\frac{\phi^3(x^3)}{\phi(x)}$

$1 + 6\left(\frac{1}{e^y - 1} - \frac{1}{e^{2y} - 1} + \frac{1}{e^{4y} - 1} - \&c\right)$

$= \frac{\phi^3(e^{-3y})}{\phi(e^{-y})}(1 + 4p + p^2)$

Cor. $\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&c$

$= \frac{1}{2x(x+1)+1} + \frac{1}{24(x^2+x)^3 + 60(x^2+x)^2 + \frac{72}{5}(x^2+x) + \frac{\phi(x)}{40}}$

$\phi(0) = 178$, $\phi(\frac{1}{2}) = 571$, $\phi(1) = 1015$, $\phi(1\frac{1}{2}) = 1384$, $\phi(2) = 1679$,

$\phi(2\frac{1}{2}) = 1916$, $\phi(3) = 2093$ nearly and $\phi(\infty) = 2880$.

If h is a positive proper fraction, then

$\frac{\phi(x+h) - \phi(x)}{\phi(x+1) - \phi(x)} = \frac{3h\,\phi(1)}{2\phi(1) + \frac{h}{2}\{3\phi(1) - 2\phi(2)\}}$ nearly

28. A series of the form $A_0 + (A_1 + A_{-1}) + (A_2 + A_{-2}) + \&c$ is called a perfect series. Hence we see that the series $A_0 + A_1 + A_2 + A_3 + \&c$ is only perfect when $A_{-1}, A_{-2}, \&c$ are all equal to 0. Thus $1 + \frac{x}{1} + \frac{x^2}{2} + \&c$ is perfect [illegible]

If $\phi(0) + \phi(1) + \phi(2) + \phi(3) + \&c$ be a perfect series, then

$\phi(x) + \{\phi(x+1) + \phi(x-1)\} + \{\phi(x+2) + \phi(x-2)\} + \&c$

$= \phi(y) + \{\phi(y+1) + \phi(y-1)\} + \{\phi(y+2) + \phi(y-2)\} + \&c$ } for all [illegible] values of x [illegible]

29. $A_0 + A_1 + A_2 + \ldots + A_n$

$= (A_n + A_{n-1} + A_{n-2} + \&c$ ad inf.)

$- (A_{-1} + A_{-2} + A_{-3} + \&c$ ad inf.) for [illegible]

We also know that $A_0 + A_1 + \ldots + A_n$

$= (A_0 - A_{n+1}) + (A_1 - A_{n+2}) + (A_2 - A_{n+3}) + \&c$ [illegible]

Cor. $\phi(0) + \frac{n}{1}\phi(1) + \frac{n(n-1)}{1\cdot 2}\phi(2) + \&c$ ad inf.

$= \phi(n) + \frac{n}{1}\phi(n-1) + \frac{n(n-1)}{1\cdot 2}\phi(n-2) + \&c$ ad inf.

Ex. If either or both of m, n be [illegible] [illegible]

$1 + \frac{mn}{\ldots}x + \frac{m(m-1)\ldots}{(n+1)(n+2)}\ldots + \ldots$ [illegible]

where [illegible] and [illegible]

$$x\psi^3(x)\psi(x^5) - 5x^2\psi(x)\psi^3(x^5)$$

$$= \frac{x}{1-x^2} - \frac{2x^2}{1-x^4} - \frac{3x^3}{1-x^6} + \frac{4x^4}{1-x^8} + \frac{6x^6}{1-x^{12}}$$

$$5\phi(x)\phi^3(x^5) - \phi^3(x)\phi(x^5)$$

$$= 4\left\{1 + \frac{x}{1+x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6}\right.$$

$$\left. - \frac{7x^7}{1+x^7} - \frac{8x^8}{1-x^8} + \frac{9x^9}{1+x^9} + \&c\right\}$$

$$25\phi(x)\phi^3(x^5) - \frac{\phi^5(x)}{\phi(x^5)}$$

$$= 24 + 40\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9} \&c.\right)$$

$$\frac{\psi^5(x)}{\psi(x^5)} - 25x^2\psi(x)\psi^3(x^5)$$

$$= 1 + 5\left(\frac{x}{1+x} - \frac{2x^2}{1+x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} + \&c\right)$$

$$\frac{\dfrac{\phi^5(x)}{\phi(x^5)} + 4 \cdot \dfrac{\psi^5(x)}{\psi(x^5)}}{\phi(x)\phi^3(x^5) + 4x^2\psi(x)\psi^3(x^5)} = 5\frac{\phi^2(x)}{\phi^2(x^5)}$$

values of m, n & x,

$$\theta = \frac{n}{(m+1)x+1-n-}\,\frac{1(1-n)(1+x)}{(m+2)x+3-n-}\,\frac{2(1-n)(1+x)}{(m+3)x+5-n-\ \&c}$$

30. $1 - \frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \&c$

$$= \frac{e^x\,\lfloor n}{x^n} - \frac{x}{x+}\,\frac{1-n}{1+}\,\frac{1}{x+}\,\frac{2-n}{1+}\,\frac{2}{x+}\,\frac{3-n}{1+\ \&c}$$

$$= \frac{e^x\,\lfloor n}{x^n} - \frac{x}{x+1-n-}\,\frac{1(1-n)}{x+3-n-}\,\frac{2(2-n)}{x+5-n-}\,\frac{3(3-n)}{x+7-n-\ \&c}$$

Cor. $\frac{1}{n} - \frac{x}{1!}\cdot\frac{1}{n+1} + \frac{x^2}{2!}\cdot\frac{1}{n+2} - \frac{x^3}{3!}\cdot\frac{1}{n+3} + \&c$

$$= \frac{\lfloor n-1}{x^n} - \frac{e^{-x}}{x+}\,\frac{1-n}{1+}\,\frac{1}{x+}\,\frac{2-n}{1+\ \&c} = \frac{\lfloor n-1}{x^n} - \frac{e^{-x}}{x+1-n-}\,\frac{1(1-n)}{x+3-n-\ \&c}$$

Ex. 1. $x - \frac{x^3}{3\,\lfloor 1} + \frac{x^5}{5\,\lfloor 2} - \&c = \frac{1}{2}\sqrt{\pi}$ when $x = \infty$

2. $x - \frac{x^3}{3^2\,\lfloor 1} + \frac{x^5}{5^2\,\lfloor 2} - \&c = \sqrt{\frac{\pi}{4}}\left(\frac{C_0}{2} + \log 2x\right)$ when $x = \infty$ [illegible]

31. $1 + \frac{x}{1.3} + \frac{x^2}{1.3.5} + \frac{x^3}{1.3.5.7} + \frac{x^4}{1.3.5.7.9} + \&c$

$$= \sqrt{\frac{\pi}{2x}e^x} - \frac{1}{x+}\,\frac{1}{1+}\,\frac{2}{x+}\,\frac{3}{1+}\,\frac{4}{x+\ \&c}$$

$$= \sqrt{\frac{\pi}{2x}e^x} - \frac{1}{x+1-}\,\frac{1.2}{x+5-}\,\frac{3.4}{x+9-}\,\frac{5.6}{x+13-\ \&c}$$

Cor. $\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+\ \&c} = \sqrt{\frac{\pi e}{2}} - \left(1 + \frac{1}{1.3} + \frac{1}{1.3.5} + \ldots\right)$

32. $\frac{x}{1\,\lfloor 1} - \frac{x^2}{2\,\lfloor 2} + \frac{x^3}{3\,\lfloor 3} - \frac{x^4}{4\,\lfloor 4} + \&c = C_0 + \log x + \frac{e^{-x}}{x+}\,\frac{1}{1+}\,\frac{1}{x+}\,\frac{2}{1+}\,\frac{2}{x+}\,\frac{3}{1+\ \&c}$

$+ \log x + e^{-x}\varphi(x)$ [illegible]

$$\varphi(x) = \frac{1}{x+1-}\,\frac{1^2}{x+3-}\,\frac{2^2}{x+5-\ \&c}$$

$\varphi(x+1) = x\{\varphi(x)\}^2$ [illegible] when x is great

$\varphi(x) - \{\varphi(x)\}^2 = \frac{1}{170}$ [illegible]

$$\varphi(x) = \frac{1}{x+1} + \frac{1}{x^4 + 6x^2 + 12x + 8}$$ where [illegible] than x

$$\frac{\psi(x^7)\psi(x^9) - \psi(-x^7)\psi(-x^9)}{\psi(x)\psi(x^{63}) - \psi(-x)\psi(-x^{63})} = x^6$$

$$\frac{\psi(x^5)\psi(x^{11}) - \psi(-x^5)\psi(-x^{11})}{\psi(x)\psi(x^{55}) + \psi(-x)\psi(-x^{55})} = x^5$$

$$\frac{\psi(x^3)\psi(x^{13}) - \psi(-x^3)\psi(-x^{13})}{\psi(x)\psi(x^{39}) + \psi(-x)\psi(-x^{39})} = x^3$$

$$x\psi^5(x)\psi(x^3) - 9x^2\psi(x)\psi^5(x^3)$$

$$= \frac{x}{1-x^2} - \frac{2^2x^2}{1-x^4} + \frac{4^2x^4}{1-x^8} - \frac{5^2x^5}{1-x^{10}} + \&c$$

$$9\phi(x)\phi^5(x^3) - \phi^5(x)\phi(x^3)$$

$$= 8\left(1 + \frac{x}{1+x} - \frac{2^2x^2}{1-x^2} + \frac{4^2x^4}{1-x^4} - \frac{5^2x^5}{1+x^5} + \frac{7^2x^7}{1+x^7} - \&c\right)$$

If $n = \frac{1}{14-\theta}$ then $x = 100n - 8.6 + \frac{.81}{9n+.14}$ very nearly

If $x > 8$, $x = \frac{.08\theta^2 + 6.34\theta - 4.45}{14-\theta}$ to 2 places of decimals

and if $x < 7$, $14 - \theta = \frac{1000}{71x+75}$ to a place of decimal

$x = 11.14$ when $\theta = 10$ and $\theta = 5.6$ when $x = 4$.

$\phi(x) = \frac{1}{x} - \frac{\lfloor 1}{x^2} + \frac{\lfloor 2}{x^3} - \cdots \pm \frac{1}{x^n}\left(\frac{\lfloor n}{x+n+1} - \frac{1(1+n)}{x+n+3} - \frac{2(2+n)}{x+n+5} - \&c\right)$

Cor. $\frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2}\left(1+\frac{1}{2}\right) + \frac{x^3}{\lfloor 3}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \&c = e^x(P_0 + \log_e x) + \phi(x)$

33. $\frac{x}{1^n\lfloor 1} - \frac{x^2}{2^n\lfloor 2} + \frac{x^3}{3^n\lfloor 3} - \frac{x^4}{4^n\lfloor 4} + \&c = \phi_n(x) + (-1)^{n-1}\psi_n(x)e^{-x}$

where $\phi_n(x)$ is the term independent of p in $\frac{x^p\lfloor -p}{p^n}$, and

$$\psi_n(x) - \psi_n'(x) = \frac{\psi_{n-1}(x)}{x}$$

$\phi_n(x) = \frac{1}{\lfloor n}\left\{A_0(\log_e x)^n + nA_1(\log_e x)^{n-1} + \frac{n(n-1)}{\lfloor 2}A_2(\log_e x)^{n-2} + \cdots + A_n\right\}$

where $\lfloor x = A_0 - A_1\frac{x}{\lfloor 1} + A_2\frac{x^2}{\lfloor 2} - A_3\frac{x^3}{\lfloor 3} + \&c$

$A_n = S_0 A_{n-1} + (n-1)S_2 A_{n-2} + (n-1)(n-2)S_3 A_{n-3} + \&c$

$\lfloor x = 1 - .5772156649x + .9890560173x^2 - .9074790803x^3 + .9817280985\frac{x^4}{1+\theta x}$

$\theta = 1$ very nearly. when $x = 0$, $\theta = 1.00027$; $x = 6$, $\theta = \frac{[illegible]}{88}$

$x = 1$, $\theta = \frac{51}{52}$. $x = 2$, $\theta = \frac{77}{82}$

$$\psi_n(x) = \cfrac{x}{x + \cfrac{n}{2 + \cfrac{[illegible]}{[illegible] + \cfrac{[illegible]}{[illegible] + \&c}}}}$$

Ex. $\left(\frac{x}{1^2\lfloor 1} - \frac{x^2}{2^2\lfloor 2} + \frac{x^3}{3^2\lfloor 3} - \frac{x^4}{4^2\lfloor 4} + \frac{x^5}{5^2\lfloor 5} - \&c\right)$ [illegible] $)^2$

$= \frac{\pi^2}{12}$ when x becomes infinitely great

$$1-\frac{n}{\lfloor 1}\left(\frac{1}{2}\right)^2+\frac{n(n-1)}{\lfloor 2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2-\&c$$

$$=\left(\frac{\lfloor n-\frac{1}{2}}{\lfloor -\frac{1}{2}\lfloor n}\right)^2\left\{1+\frac{1^2}{2\cdot 4}\cdot\frac{n(n+\frac{1}{2})}{(n-\frac{1}{2})^2}+\frac{1^2\cdot 5^2}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{n(n-1)\cdot(n+\frac{1}{2})(n-\frac{1}{2})}{(n-\frac{1}{2})^2(n-\frac{3}{2})^2}+\&c\right\}$$

$$\frac{\pi}{4}\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{1+x}{2}\right)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1+x}{2}\right)^2+\&c\right\}^2$$

$$-\frac{\pi}{4}\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{1-x}{2}\right)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1-x}{2}\right)^2+\&c\right\}^2$$

$$=x+\frac{x^3}{2}+\frac{41x^5}{120}+\frac{21x^7}{80}+\&c$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{x\sin\theta\,d\theta\,d\phi}{(1-x^2\sin^2\theta)\sqrt{1+x^2\sin^2\theta\sin^2\phi}}=\phi(x)$$

$$\phi(1-2x^2) = \frac{5}{(1+x)}\int\left\{2\left(\frac{1-x^2}{1+x}\right)^2-1\right\}$$

$$=\frac{4}{(1+x)^4}\int\left\{2\left(\frac{1-x}{1+x}\right)^4-1\right\}$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{x\sin\theta\,d\theta\,d\phi}{\sqrt{1-x^2\sin^2\theta}\sqrt{1-x^2\sin^2\theta\sin^2\phi}}$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x}\frac{d\theta}{\sqrt{1-x^2\sin^2\phi-\sin^2\theta\cos^2\phi}}\,d\phi.$$

CHAPTER XV

1. $\left[1+\left(\frac{1}{2}\right)^2\left\{1-\frac{4\varphi(x)}{\varphi(x)}\right\}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left\{1-\frac{4\varphi(x)}{\varphi(x)}\right\}^2+\&c\right]\frac{1}{\sqrt{\varphi(x)}}$

is always an even function of x

2. $1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2(1-x)^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2(1-x)^3+\&c$

$=$ [illegible] $\left\{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2(1-x)^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2(1-x)^3+\&c\right\}$

3. $1+\left(\frac{1}{2}\right)^2\left(\frac{4x}{1+2x+x^2}\right)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{4x}{1+2x+x^2}\right)^2+\&c$

$=(1+x)\left\{1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2x^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2x^6+\&c\right\}$

Cor. $1+\left(\frac{1}{2}\right)^2\left\{\frac{8x(1+x^2)}{(1+x)^4}\right\}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left\{\frac{8x(1+x^2)}{(1+x)^4}\right\}^2+\&c$

$=(1+x)^2\left\{1+\left(\frac{1}{2}\right)^2x^4+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2x^8+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2x^{12}+\&c\right\}$

4. $1+\left(\frac{1}{2}\right)^2\frac{2x}{1+x}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{2x}{1+x}\right)^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\frac{2x}{1+x}\right)^3+\&c$

$=\sqrt{1+x}\left(1+\frac{1\cdot 3}{4^2}x^2+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}x^4+\&c\right)$

5. $1+\left(\frac{1}{4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$

$=1+\left(\frac{1}{4}\right)^2x+\left(\frac{1\cdot 5}{4\cdot 8}\right)^2x^2+\left(\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}\right)^2x^3+\&c$

6. $1+\left(\frac{1}{2}\right)^3x+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3x^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3x^3+\&c$

$=\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}^2$

7. Let $\pi\alpha\beta=1$ and $\alpha=\frac{\sqrt{\pi}}{\left(\frac{1}{4}\right)^2}$ [illegible] such that

$\alpha=1.180340,599016,092$ ✓

$\beta=.269676,300574,191$ ✓

$1/\beta=3.708144,354602,131$ ✓

$1+\left(\frac{1}{2}\right)^2\left(\frac{1+x}{2}\right)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1+x}{2}\right)^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\frac{1+x}{2}\right)^3+\&c$

$=\alpha\left\{1+\frac{1^2}{3\cdot 4}x^2+\frac{1^2\cdot 5^2}{[illegible]}x^4+\frac{1^2\cdot 5^2\cdot 9^2}{[illegible]\cdot 10\cdot 12}x^6+\&c\right\}$

$+\beta\left\{x+\frac{3^2}{4\cdot 5}x^3+\frac{3^2\cdot 7^2}{4\cdot 6\cdot 8\cdot 10}x^5+\frac{3^2\cdot 7^2\cdot 11^2}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}x^7+\&c\right\}$

8. There are [illegible] solutions in the form [illegible]

$$\psi(x)\psi(x^5) - \psi(-x)\psi(-x^5)$$
$$= 2x\, f(x^2, x^4)\, f(x^{40}, x^{80}) + 4x^{15}\psi(x^6)\psi(x^{120})$$

$$\psi(p)\psi(q) = \psi(pq) + p f\left(\frac{q}{p}, p^2\right) + p^3 q f\left(\frac{q^2}{p^2}, \frac{p^3}{q}\right)$$
$$+ p^6 q^3 f\left(\frac{q^3}{p^3}, \frac{p^5}{q^2}\right) + \&c$$

$$\phi(p)\phi(q) = \phi(pq) + 2p f\left(\frac{q}{p}, p^3 q\right) +$$
$$2pq f\left(\frac{p^3}{q}, \frac{q^3}{p}\right) + 2p^4 q f\left(\frac{q^3}{p^3}, \frac{p^5}{q}\right) +$$
$$2p^4 q^4 f\left(\frac{p^5}{q^3}, \frac{q^5}{p^3}\right) + 2p^9 q^4 f\left(\frac{q^5}{p^5}, \frac{p^7}{q^3}\right)$$
$$+ 2p^9 q^9 f\left(\frac{p^7}{q^5}, \frac{q^7}{p^5}\right) + \&c$$

$$\sqrt[i]{1+x^2}\left\{1 + \left(\frac{i}{2}\right)^2 \frac{1 \pm \ell x}{2} + \&c\right\}$$
$$= \frac{1+i}{2}\left\{1 + \left(\frac{i}{2}\right)^2 \frac{1 + \frac{x}{\sqrt{1+x^2}}}{2} + \&c\right\}$$
$$+ \frac{1-i}{2}\left\{1 + \left(\frac{i}{2}\right)^2 \frac{1 - \frac{x}{\sqrt{1+x^2}}}{2} + \&c\right\}.$$

18. $1+\left(\frac{1}{2}\right)^2\left\{\frac{(1+x)^2}{2(1+x^2)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{(1+x)^2}{2(1+x^2)}\right\}^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left\{\frac{(1+x)^2}{2(1+x^2)}\right\}^3+\&c$

$=$ [illegible] $\sqrt{1+x^2}\left(1+\frac{1}{2}\cdot\frac{1}{3}x^4+\frac{1.3}{2.4}\cdot\frac{1.5}{3.7}x^8+\frac{1.3.5}{2.4.6}\cdot\frac{1.5.9}{3.7.11}x^{12}+\&c\right)$

$+$ [illegible] $x\sqrt{1+x^2}\left(1+\frac{1}{2}\cdot\frac{3}{5}x^4+\frac{1.3}{2.4}\cdot\frac{3.7}{5.9}x^8+\frac{1.3.5}{2.4.6}\cdot\frac{3.7.11}{5.9.13}x^{12}+\&c\right)$

Ex. 1. $1+\left(\frac{1}{2}\right)^2\frac{x}{1+x}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{x}{1+x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{x}{1+x}\right)^3+\&c$

$=\sqrt{\frac{1+x}{1-x}}\left\{1-\frac{1.3}{4^2}\left(\frac{4x}{1-2x+x^2}\right)+\frac{1.3.5.7}{4^2.8^2}\left(\frac{4x}{1-2x+x^2}\right)^2-\&c\right\}$

2. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$=\frac{1}{\sqrt{1+x}}\left[1+\frac{1.3}{4^2}\left\{\frac{4x}{(1+x)^2}\right\}+\frac{1.3.5.7}{4^2.8^2}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c\right]$

3. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$=\frac{1}{\sqrt{1-x}}\left[1-\left(\frac{1}{4}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}+\left(\frac{1.5}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c\right]$

4. $1+\left(\frac{1}{2}\right)^2\frac{1-\sqrt{1-x}}{2}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$

$=\sqrt{1-x}\left\{1+\left(\frac{3}{4}\right)^2x+\left(\frac{3.7}{4.8}\right)^2x^2+\left(\frac{3.7.11}{4.8.12}\right)^2x^3+\&c\right\}$

5. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$=\frac{(1+x)}{(1-x)\sqrt{1-x}}\left[1-\left(\frac{3}{4}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}+\left(\frac{3.7}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c\right]$

6. $1+\left(\frac{1}{2}\right)^2\frac{1+x}{2}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\&c$

$=\frac{\alpha}{\sqrt[4]{1-x^2}}\left\{1-\frac{1^2}{2.4}\cdot\frac{x^2}{1-x^2}+\frac{1^2.5^2}{2.4.6.8}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$

$+$ [illegible] $\cdot\frac{x}{\sqrt[4]{1-x^2}}\left\{1-\frac{3^2}{4.6}\cdot\frac{x^2}{1-x^2}+\frac{3^2.7^2}{4.6.8.10}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$

7. $1+\left(\frac{1}{2}\right)^2\left\{\frac{(1+x)^2}{2(1+x^2)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{(1+x)^2}{2(1+x^2)}\right\}^2+\&c$

$=\frac{\alpha}{\sqrt{1-x^2}}\left\{1+\frac{1}{2}\cdot\frac{1}{3}\left(\frac{x^4}{1-x^4}\right)+\frac{1.3}{2.4}\cdot\frac{1.5}{3.7}\left(\frac{x^4}{1-x^4}\right)^2+\&c\right\}$

$+\frac{2\alpha x}{[illegible]}\left\{1+[illegible]\left(\frac{x^4}{1-x^4}\right)+\ldots\right\}$

$$x\psi^2(x)\psi^2(x^3)$$
$$= \frac{x}{1-x^2} + \frac{2x^2}{1-x^4} + \frac{4x^4}{1-x^8} + \frac{5x^5}{1-x^{10}} + \&c$$

$$\phi^2(x)\phi^2(x^3)$$
$$= 1 + 4\Big(\frac{x}{1-x} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{7x^7}{1-x^7} + \frac{8x^8}{1-x^8}$$
$$+ \&c\Big)$$

$$x\psi(x^2)\psi(x^6)$$
$$= \frac{x}{1-x^2} - \frac{x^5}{1-x^{10}} + \frac{x^7}{1-x^{14}} - \frac{x^{11}}{1-x^{22}} + \frac{x^{13}}{1-x^{26}} - \&c$$

$$x\psi(x)\psi(x^7)$$
$$= \frac{x}{1-x} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^9}{1-x^9} + \frac{x^{11}}{1-x^{11}} - \frac{x^{13}}{1-x^{13}}$$
$$+ \frac{x^{15}}{1-x^{15}} - \frac{x^{17}}{1-x^{17}} - \frac{x^{19}}{1-x^{19}} + \frac{x^{23}}{1-x^{23}} \&c$$

$$\psi(x^3)\psi(x^5) - \psi(-x^3)\psi(-x^5) = 2x^3\psi(x^2)\psi(\dots$$
$$\psi(x)\psi(x^3) - \psi(-x)\psi(-x^3) = 2x\phi(x^2)\psi(\dots$$

$$\psi(x)\psi(x^{11}) - \psi(-x)\psi(-x^{11})$$
$$= 2xf(x^2, x^{10})f(x^{44}, x^{88}) + 2x^{15}\phi(x^6)\psi(x^{132}).$$

$$\phi(x)\phi(x^7) = 1 + 2\Big(\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^4}{1-x^4}$$
$$- \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} + \frac{x^8}{1-x^8} + \frac{x^9}{1-x^9} - \frac{x^{10}}{1-x^{10}}$$

$= 2\sqrt{2} \left\{1 + \left(\frac{1}{2}\right)^2 (\sqrt{2}-1)^2 + \left(\frac{1.3}{2.4}\right)^2 (\sqrt{2}-1)^4 + \ldots\right\}$ [illegible]

$\ldots + \left(\frac{1.3}{2.4}\right)^2 \left(\ldots\right)^2 + \&c = \frac{1}{\sqrt{\ldots}} \left\{1 + \frac{1.3}{4^2} \cdot \frac{\ldots}{\ldots} + \frac{1.3.5.7}{4^2.8^2} \cdot \frac{\ldots}{\ldots}\right.$

9\. $1 + \left(\frac{1}{2}\right)^2 \ldots + \left(\frac{1.3}{2.4}\right)^2 \left(\ldots\right)^2 + \&c = \frac{\ldots}{\sqrt{\ldots}} \left\{1 + \frac{1.3}{4^2} \cdot \frac{\ldots}{\ldots} + \frac{1.3.5.7}{4^2.8^2} \cdot \frac{\ldots}{\ldots}\right.$

10\. $1 + \left(\frac{1}{2}\right)^2 \left(\ldots\right) + \left(\frac{1.3}{2.4}\right)^2 \left(\ldots\right)^2 + \&c$

11\. $1 + \left(\frac{1}{2}\right)^2 \frac{1}{3} + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{5} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{7} + \&c$

$= \dfrac{1 - \frac{1}{4} + \frac{\ldots}{\ldots} - \frac{\ldots}{\ldots} + \&c}{1 - \frac{\ldots}{3} + \frac{\ldots}{5} - \frac{\ldots}{7} + \&c}$

12\. $1 + \left(\frac{1}{2}\right)^2 \frac{1}{5} + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{9} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{13} + \&c = \frac{\ldots}{(\ldots)^2 \sqrt{\ldots}}$ [illegible]

13\. $1 - \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 - \left(\frac{1.3.5}{2.4.6}\right)^3 + \&c = \left(\frac{\ldots}{\ldots}\right)^2$

$1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \&c = \frac{2}{\pi}$

14\. $1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1.5}{4.8}\right)^2 + \left(\frac{1.5.9}{4.8.12}\right)^2 + \&c = \frac{\sqrt{\pi}}{(\ldots)^2}$ [illegible]

15\. $1 - \left(\frac{1}{4}\right)^2 + \left(\frac{1.5}{4.8}\right)^2 - \left(\frac{1.5.9}{4.8.12}\right)^2 + \&c =$ [illegible]

16\. $1 + \left(\frac{1}{4}\right)^3 + \left(\frac{1.5}{4.8}\right)^3 + \left(\frac{1.5.9}{4.8.12}\right)^3 + \&c = \frac{\sqrt{\pi}}{\ldots (\ldots)^2 \sqrt{2}}$

17\. $1 - 9\left(\frac{1}{4}\right)^3 + 17\left(\frac{1.5}{4.8}\right)^3 - 25\left(\frac{1.5.9}{4.8.12}\right)^3 + \&c = \frac{2\sqrt{2}}{\sqrt{\pi}}$

18\. $1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1.5}{4.8}\right)^4 + 25\left(\frac{1.5.9}{4.8.12}\right)^4 + \&c =$ [illegible]

20\. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi}{\sqrt{1 - \sin^2\theta \sin^2\phi \sin^2\psi}} = \int_0^{\frac{\pi}{2}} \frac{d\ldots}{\sqrt{1 - \ldots}}$

21\. $\displaystyle\int_0^{\frac{\pi}{2}} \left\{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \sin^2\theta \sin^2\phi}}\right\}^2 d\phi = \frac{\pi^3}{16}\left\{1 + \left(\frac{1}{2}\right)^4 + \left(\frac{1.3}{2.4}\right)^4 + \ldots\right.$

9\. If $\phi(x)$ can be expressed in n different ways the nth way being $U_n + V_n$ and if [illegible] [illegible] and V_1, V_2 [illegible] must be identically equal then [illegible]

$$+ \phi(n) \lfloor a+n \lfloor b+n \lfloor -n-1 - \frac{\phi(n+1)}{\lfloor 1} \lfloor a+n+1 \lfloor b+n+1 \lfloor -n-2 + \&c$$

$$\lfloor a+n \lfloor b+n \left\{ \phi(0) \frac{\lfloor a \lfloor b}{\lfloor a+b+n+1} + \frac{\phi(0)-\phi(1)}{\lfloor 1} \frac{\lfloor a+1 \lfloor b+1}{\lfloor a+b+n+2} \right.$$

$$\left. + \frac{\phi(0) - 2\phi(1) + \phi(2)}{\lfloor 2} \cdot \frac{\lfloor a+2 \lfloor b+2}{\lfloor a+b+n+2} + \&c \right\}$$

The irreducible part in $1 - 24\left\{ \frac{y}{(1-y)^2} \cos 2\theta + \frac{y^2}{(1-y^2)^2} \cos 4\theta + \&c \right\} = \frac{\theta^2}{6}\left\{1 - 24\left(\frac{y}{1-y} + \frac{2y^2}{1-y^2} + \&c\right)\right\}^2$

$$+ \theta \left\{1 - 24\left(\frac{y}{1-y} + \frac{2y^2}{1-y^2} + \&c\right)\right\}\left\{\cot\theta + 4\left(\frac{y}{1-y}\sin 2\theta + \frac{y^2}{1-y^2}\sin 4\theta + \&c\right)\right\}.$$

$$Q_n = \phi(n) + \frac{n+1}{\underline{|1}}\phi(n+1) + \ldots$$

$$\phi(0) + (1-x)\phi(1) + (1-x)^2\phi(2) + (1-x)^3\phi(3) + \&c$$

$$= Q_0 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c$$

$$+ \frac{1}{\left(\log_e \frac{1}{1-x}\right)^{n+1}}\left\{P_0 + P_1 \log_e \frac{1}{1-x} + P_2\left(\log_e \frac{1}{1-x}\right)^2 + P_3\left(\log_e \frac{1}{1-x}\right)^3 + \&c\right\}$$

Cor. 1. If $$Q'_n = \frac{\underline{|m}}{\underline{|m-n}\,\underline{|n}}\phi(m) + \frac{\underline{|m+1}}{\underline{|m-n+1}\,\underline{|n}}\phi(m+1) + \frac{\underline{|m+2}}{\underline{|m-n+2}\,\underline{|n}}\phi(m+2) + \&c$$

then $$\phi(m)(1-x)^m + \phi(m+1)(1-x)^{m+1} + \phi(m+2)(1-x)^{m+2} + \&c$$

$$= Q'_0 - Q'_1 x + Q'_2 x^2 - Q'_3 x^3 + \&c$$

$$+ \frac{1}{\left(\log_e \frac{1}{1-x}\right)^{n+1}}\left\{P_0 + P_1 \log_e \frac{1}{1-x} + P_2\left(\log_e \frac{1}{1-x}\right)^2 + \&c\right\}$$

Cor 2. If $a+b+c+1 = d+e$; then

$$\left\{\frac{\underline{|a}\,\underline{|b}\,\underline{|c}}{\underline{|d}\,\underline{|e}} + \frac{1-x}{\underline{|1}}\cdot\frac{\underline{|a+1}\,\underline{|b+1}\,\underline{|c+1}}{\underline{|d+1}\,\underline{|e+1}} + \frac{(1-x)^2}{\underline{|2}}\frac{\underline{|a+2}\,\underline{|b+2}\,\underline{|c+2}}{\underline{|d+2}\,\underline{|e+2}} + \&c\right.$$

$$+ \log_e x \quad \text{when } x = 0$$

$$= -\Sigma\frac{1}{a} - \Sigma\frac{1}{b} + 1\cdot\frac{(c-d)(c-e)}{(a+1)(b+1)} + \frac{1}{2}\cdot\frac{(c-d)(c-d-1)\ldots}{(a+1)(a+2)\ldots(b+1)\ldots} \ldots$$

$$11.\ \underline{|a}\,\underline{|b}\left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|a+b+n+1}} + \frac{1-x}{\underline{|1}}\cdot\frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|a+b+n+2}} + \&c\right\}.$$

$$= \left\{\underline{|a+n}\,\underline{|b+n}\,\underline{|-n-1} - \frac{x}{\underline{|1}}\underline{|a+n+1}\,\underline{|b+n+1}\,\underline{|-n-2} + \&c\right\}$$

$$+ \frac{1}{x^n}\left\{\underline{|a}\,\underline{|b}\,\underline{|n-1} - \frac{x}{\underline{|1}}\underline{|a+1}\,\underline{|b+1}\,\underline{|n-2} + \&c\right\}$$

This is true for all values of n.

N.B. If n is an integer R.H.S takes the form ∞ − ∞ [illegible] should write n+h [illegible] and then [illegible] h [illegible]

$$\frac{1}{\lfloor\gamma} + \frac{\alpha\beta}{\lfloor 1\,\lfloor\gamma+1}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2\,\lfloor\gamma+2}x^2 + \&c$$

$$\frac{dy}{dx} = -\frac{\lfloor\gamma-1\,\lfloor\alpha-1\,\lfloor\beta-1}{\frac{\lfloor\gamma-1}{\lfloor\alpha-1\,\lfloor\beta-1}\,x^{\gamma+1}(1-x)^{\delta}\left\{\frac{1}{\lfloor\gamma} + \frac{\alpha\beta}{\lfloor 1\,\lfloor\gamma+1}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2\,\lfloor\gamma+2}x^2\right.}$$

If $x^5 = (1+x)(1+x+x^2)$

then $x = \dfrac{e^{\frac{\pi}{24}\sqrt{47}}}{\sqrt{2}}(1+e^{-\pi\sqrt{47}})(1+e^{-3\pi\sqrt{47}})(1+e^{-5\pi\sqrt{47}})$ &c

Cor. 1. If n is a positive integer

$$\underline{|a}\,\underline{|b}\left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|a+b+n+1}} + \frac{1-x}{1} \cdot \frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|a+b+n+2}} + \&c\right\}$$

$$(n+1)^n (\log_e x)\left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|n}} + \frac{x}{1}\,\frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|n+1}} + \&c\right\}$$

$$+ (-1)^n \left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|n}}\left(x\frac{1}{a+n} + x\frac{1}{b+n} - x\frac{1}{n} - 0\right)\right.$$

$$+ \frac{x}{1}\,\frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|n+1}}\left(x\frac{1}{a+n+1} + x\frac{1}{b+n+1} - x\frac{1}{n+1} - \ldots\right)$$

$$+ \frac{x^2}{1\cdot 2}\,\frac{\underline{|a+n+2}\,\underline{|b+n+2}}{\underline{|n+2}}\left(x\frac{1}{a+n+2} + x\frac{1}{b+n+2} - \ldots \frac{1}{n+2} - x\frac{1}{2}\right)$$

$$\left. + \&c \right\}$$

$$= \frac{1}{2n}\left\{\underline{|a}\,\underline{|b}\,\underline{|n-1} - \frac{x}{1}\,\underline{|a+1}\,\underline{|b+1}\,\underline{|n-2} + \&c \text{ to } n \text{ terms}\right\}$$

Cor. 2. If n is a negative integer,

$$\underline{|a}\,\underline{|b}\left\{\frac{\underline{|a+n}\,\underline{|b+n}}{\underline{|a+b+n+1}} + \frac{1-x}{1}\,\frac{\underline{|a+n+1}\,\underline{|b+n+1}}{\underline{|a+b+n+2}} + \&c\right\}$$

$$+ (-x)^{-n} (\log_e x)\left\{\frac{\underline{|a}\,\underline{|b}}{\underline{|-n}} + \frac{x}{1}\,\frac{\underline{|a+1}\,\underline{|b+1}}{\underline{|-n+1}} + \&c\right\}$$

$$+ (-x)^{-n}\left\{\frac{\underline{|a}\,\underline{|b}}{\underline{|-n}}\left(x\frac{1}{a} + x\frac{1}{b} - x\frac{1}{-n} - 0\right)\right.$$

$$+ \frac{x}{1}\,\frac{\underline{|a+1}\,\underline{|b+1}}{\underline{|-n+1}}\left(x\frac{1}{a+1} + x\frac{1}{b+1} - \ldots\right) \text{[illegible]}$$

$$+ \frac{x^2}{1\cdot 2}\,\frac{\underline{|a+2}\,\underline{|b+2}}{\underline{|-n+2}}\left(x\frac{1}{a+2} + \ldots\right) \text{[illegible]}$$

$$\left. + \&c \right\}$$

$$= \underline{|a+n}\,\underline{|b+n}\,\underline{|-n-1} - \frac{x}{1}\,\underline{|a+n+1}\,\underline{|b+n+1} \ldots \text{[illegible]}$$

$$\lfloor a \lfloor b \left\{ \phi(0) \frac{\lfloor a \lfloor b}{\lfloor a+b+1} + \frac{\phi(0)-\phi(1)}{\lfloor 1} \cdot \frac{\lfloor a+1 \lfloor b+1}{\lfloor a+b+3} + \&c \right\}$$

$$+ \phi'(0) \lfloor a \lfloor b + \phi'(1) \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} + \phi'(2) \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} + \&c$$

$$+ \phi(0) \lfloor a \lfloor b \left(\Sigma \frac{1}{a} + \Sigma \frac{1}{b} - 2\Sigma 1\right) + \phi(1) \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} \times$$

$$\left(\Sigma \frac{1}{a+1} + \Sigma\frac{1}{b+1} - 2\Sigma\frac{1}{1}\right) + \phi(2) \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2}\left(\Sigma\frac{1}{a+2} + \Sigma\frac{1}{b+2} - 2\Sigma\frac{1}{2}\right) + \&c$$

$$= 0.$$

If $y = \dfrac{\lfloor \alpha-1 \lfloor \beta-1}{\lfloor \alpha+\beta-1} \cdot \dfrac{1 + \frac{\alpha}{1}\cdot\frac{\beta}{\alpha+\beta}(1-x) + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2 (\alpha+\beta)(\alpha+\beta+1)}(1-x)^2 + \&c}{1+\frac{\alpha}{1}\cdot\frac{\beta}{1}x + \frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\lfloor 2}x^2+\&c}$

then $\dfrac{dy}{dx} = -\dfrac{1}{x(1-x)^{\alpha+\beta}\left\{1+\frac{\alpha}{1}\cdot\frac{\beta}{1}x + \frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\lfloor 2}x^2+\&c\right\}^2}$

~~If $\alpha+\beta=\gamma$, then~~ If $\alpha+\beta+1 = \gamma+\delta$.

$$\int \frac{\int x^{n-2}\left\{1+\frac{\alpha}{1}\cdot\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c\right\}dx}{x^\gamma(1-x)^\delta\left\{1+\frac{\alpha}{1}\cdot\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c\right\}^2}dx$$

$$= \frac{x^{n-\gamma}(1-x)^{1-\delta}}{(n-\gamma)(n-1)}\left\{1 + \frac{(n-\alpha)(n-\beta)}{n(n-\gamma+1)}x + \frac{(n-\alpha)(n-\alpha+1)(n-\beta)(n-\beta+1)}{n(n+1)(n-\gamma+1)(n-\gamma+2)}x^2 + \&c\right\}$$

$$\div\left\{1+\frac{\alpha}{1}\cdot\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \right\}$$

[illegible]

$+ \ldots \Big\{ \lfloor a \lfloor b + x \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} + x^2 \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} + \&c \Big\}$

$+ \lfloor a \lfloor b \left(\Sigma \frac{1}{a} + \Sigma \frac{1}{b} - 2\Sigma 1 \right) + x \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} \left(\Sigma \frac{1}{a+1} + \Sigma \frac{1}{b+1} - 2\Sigma 1 \right)$

$+ x^2 \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} \left(\Sigma \frac{1}{a+2} + \Sigma \frac{1}{b+2} - 2\Sigma \frac{1}{2} \right) + \&c \Big\}$

$= 0.$

Cor. $\pi \left\{ 1 + \left(\frac{1}{2}\right)^2 (1-x) + \left(\frac{1.3}{2.4}\right)^2 (1-x)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 (1-x)^3 + \&c \right\}$

$= \left(\log \frac{16}{x}\right) \left\{ 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c \right\}$

$- 4 \left\{ \left(\frac{1}{2}\right)^2 \frac{1}{1.2} x + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{1}{1.2} + \frac{1}{3.4}\right) x^2 + \&c \right\}$

Ex. 1. $4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\tan \frac{\theta}{2}}{\sqrt{1 - k^2 \cos^2\theta \cos^2\phi}} \, d\theta \, d\phi$

$= \pi \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-k^2)\sin^2\theta}} + \log k \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$

2. $\left\{ \frac{\lfloor a \lfloor b}{\lfloor a+b+1} + \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor a+b+2} + \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor a+b+3} + \&c \text{ to } n \text{ terms} \right\}$

$- \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c \text{ to } n \text{ terms} \right\}$ when $n = \infty$

$= - \Sigma \frac{1}{a} - \Sigma \frac{1}{b}$

3. $\pi \left\{ 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c \right\} - \left(1 + \frac{1}{2} + \frac{1}{3} + \&c\right) =$ [illegible]

4. If $\phi(n) = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \&c$ to n terms, then

$\pi \phi(n) = 5 \log 2 + \ldots + \frac{1}{4n} - \ldots$ [illegible]

$= 5 \log 2 + \ldots$ [illegible]

[illegible]

If $\alpha+\beta+1=\gamma+\delta$

$$\text{and } \phi(x) = \frac{1+\frac{\alpha}{1}\cdot\frac{\beta}{\delta}(1-x)+\frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}(1-x)^2+\&c}{1+\frac{\alpha}{1}\cdot\frac{\beta}{\gamma}x+\frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c},$$ then

$$\phi'(x) = -\frac{\lfloor\gamma-1\,\lfloor\delta-1}{\lfloor\alpha-1\,\lfloor\beta-1}\cdot\frac{1}{x^\gamma(1-x)^\delta\left\{1+\frac{\alpha}{1}\cdot\frac{\beta}{\gamma}x+\frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c\right\}^2}$$

$$\left\{1+\frac{\alpha}{1}\cdot\frac{\beta}{\gamma}\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}$$

$$\times\left\{1+\frac{\alpha}{1}\cdot\frac{\beta}{\delta}\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}$$

$$=1+\frac{\alpha}{\gamma}\cdot\frac{\beta}{\delta}\cdot\frac{\alpha+\beta+1}{2}\cdot\frac{\alpha+\beta}{2\alpha+2\beta}x+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\cdot\frac{(\alpha+\beta+1)(\alpha+\beta+3)}{2\cdot 4}\frac{(\alpha+\beta)(\alpha+\beta+\ldots)}{(2\alpha+2\beta)(2\alpha+\ldots)}$$

$$\times x^2+\&c$$

iii. $\phi(\tfrac{1}{2}) = \tfrac{1}{2}$

iv. $\phi(1) = \dfrac{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c}{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c}$

v. $\phi(n+1) = \phi(n) + \dfrac{1}{\pi}\left(\dfrac{\lfloor n - \tfrac{1}{2}}{\lfloor n}\right)^2$

vi. $1 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{2 \cdot 4}{3 \cdot 5}\right)^2 + \left(\tfrac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\right)^2 + \&c$ to n terms $= \dfrac{\pi^2}{4}\phi(n + \tfrac{1}{2})$

$- 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right)$

vii. $1 + (4n+3)^2\left\{\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2 \ldots\right\}$ — $1 + \left(\tfrac{3}{5}\right)^2 + \left(\tfrac{3 \cdot 7}{5 \cdot 9}\right)^2 + \left(\tfrac{3 \cdot 7 \cdot 11}{5 \cdot 9 \cdot 13}\right)^2 + \&c$ to n terms$\}$ $= 2\phi(n + \ldots$

14. $\dfrac{a}{1-m} + \dfrac{(a+b)mn}{1-mx} + \dfrac{(a+2b)m^2 n^2}{1-mx^2} + \dfrac{(a+3b)m^3 n^3}{1-mx^3} + \&c$

$= a\dfrac{1-mn}{(1-m)(1-n)} + (a+b)\dfrac{1-mnx^2}{(1-mx)(1-nx)}(mnx) + (a+2b)\dfrac{1-mnx^4}{(1-mx^2)(1-nx^2)}$

$\times (mnx^2)^2 + (a+3b)\dfrac{1-mnx^6}{(1-mx^3)(1-nx^3)}(mnx^3)^3 + \&c$

$+ \dfrac{b}{m}\left\{\dfrac{mn}{(1-n)^2} + \dfrac{(mnx)^2}{(1-nx)^2} + \dfrac{(mnx^2)^3}{(1-nx^2)^2} + \dfrac{(mnx^3)^4}{(1-nx^3)^2} + \ldots\right\}$

Cor. $\dfrac{a}{1-m} + \dfrac{(a+b)m}{1-mx} + \dfrac{(a+2b)m^2}{1-mx^2} + \&c$

$= a\dfrac{1+m}{1-m} + (a+b)\dfrac{1+mx}{1-mx}(m^2x) + (a+2b)\dfrac{1+mx^2}{1-mx^2}(m^4x^4) + \&c$

$+ b\left\{\dfrac{m}{(1-m)^2} + \dfrac{m^3x^2}{(1-mx)^2} + \dfrac{m^5x^6}{(1-mx^2)^2} + \dfrac{m^7x^{12}}{(1-mx^3)^2} + \&c\right\}$

Cor. 2. If ϕ_n denote the no. of [illegible] in the [illegible]

then $\dfrac{\phi_1}{x} + \dfrac{\phi_2}{x^2} + \dfrac{\phi_3}{x^3} + \dfrac{\phi_4}{x^4} + \&c$

$= \dfrac{1}{x-1} + \dfrac{1}{x^2-1} + \dfrac{1}{x^3-1} + \dfrac{1}{x^4-1} + \&c$

hence show that $\dfrac{1}{x-1} + \dfrac{1}{x^2-1} + \dfrac{1}{x^3-1} + \&c$ [illegible]

$= \ldots \dfrac{x+1}{x-1} \ldots$ [illegible]

Cor. 3. $\dfrac{1}{e^{2\pi}-1} + \dfrac{2}{e^{4\pi}-1} + \dfrac{3}{e^{6\pi}-1} + \&c = \ldots$ [illegible] $+ \&c$

15. $\dfrac{1+b}{1-a}, \dfrac{1+bx}{1-ax}, \dfrac{1+bx^2}{1-ax^2}, \dfrac{1+bx^3}{1-ax^3}, \&c$

[illegible] $\dfrac{a+b}{x} + \ldots \dfrac{\ldots + bx}{1-x^2} + \ldots$ [illegible]

$$f(x^3, x^6) = \psi(x) - x\psi(x^9)$$

$$(a+1)(b+1)(c+1) + (a-1)(b-1)(c-1)$$
$$= 2(a+b+c+abc)$$

$$\frac{\phi^2(x)}{\phi^2(-x)} + \frac{\phi^2(y)}{\phi^2(-y)} + \frac{\phi^2(z)}{\phi^2(-z)} + \frac{\phi^2(x)\phi^2(y)\phi^2(z)}{\phi^2(-x)\phi^2(-y)\phi^2(-z)}$$
$$= 4\frac{\phi^2(x^4)\phi^2(y^4)\phi^2(z^4)}{\phi^2(-x)\phi^2(-y)\phi^2(-z)} + 256xyz\frac{\psi^2(x^4)\psi^2(y^4)\psi^2(z^4)}{\phi^2(-x)\phi^2(-y)\phi^2(-z)}$$

$$f(-x^2, -x^3) f(-x, -x^4) = f(-x, -x^2) f(-x^5, -x^{10})$$

$$f(-x, -x^6) f(-x^2, -x^5) f(-x^3, -x^4)$$
$$= f(-x, -x^2) f^2(-x^7, -x^{14})$$

$$f(-x, -x^5) = \psi(x^3)\sqrt[3]{\frac{\phi(-x)}{\psi(x)}}$$

16. Let $f(p,q) = 1 + (p+q) + pq(p^2+q^2) + (pq)^3(p^3+q^3) + (pq)^6(p^4+q^4)$
$+ (pq)^{10}(p^5+q^5) + \&c$, then

i. $f(p,q) = f(q,p)$. ii. $f(1,p) = 2f(p,p^3)$ iii. $f(-1,p) = 0$

iv. If n is any integer, then

$$f(p,q) = p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} f\left\{p(pq)^n, q(pq)^{-n}\right\}$$

These 4 results are evident from the series itself.

v. $f(p,q) = (1+p)(1+kp)(1+k^2p)(1+k^3p)\ \&c$
$\times (1+q)(1+kq)(1+k^2q)(1+k^3q)\ \&c$
$\times (1-pq)(1-kpq)(1-k^2pq)(1-k^3pq)\ \&c$
where $k = pq$.

Sol. Since $f(-1,p) = 0$ by iii, we see from iv that if $p(pq)^n$ [illegible] or $q(pq)^n = -1$ then $f(p,q) = 0$ ∴ $(1+pk^n)$ & $(1+qk^n)$ are factors of $f(p,q)$. Again we see that if $(pq)^n = 1$ [illegible]
$f(p,q)\left\{1 - \left(\frac{p}{q}\right)^{\frac{n}{2}}\right\} = 0$ ∴ $f(p,q) = 0$ [illegible]
Hence $f(p,q) = \prod_{n=0}^{\infty}(1+pk^n)(1+qk^n)(1-pq\,k^n)$

vi. $f(p, k^{n-1}q) \times f(kp, k^{n-2}q) \times f(k^2p, k^{n-3}q) \times \ldots f(k^{n-1}p, q)$
$$= f(p,q)\frac{\left\{f(-k^n, -k^{2n})\right\}^n}{f(-k, -k^2)}$$

vii. $f(p,q) = f\left(p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}}, p^{\frac{n(n-1)}{2}} q^{\frac{n(n+1)}{2}}\right)$
$= p\, f\left\{\frac{q}{p}, p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}}\right\} + p^3 q\, f\left\{\frac{q^2}{p^2}, p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}}\right\}$
$+ p^6 q^3 f\left\{\frac{q^3}{p^3}, p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}}\right\} + \&c$ to $(n-1)$ terms.

viii. If $pq = rs$, $f(p,q)f(r,s) + f(-p,-q)f(-r,-s)$
$= 2f(ps, qr)\, f(pr, qs)$

If $pq = rs$, $f(p,q)f(r,s) - f(-p,-q)f(-r,-s)$
[illegible]

$$e^{-\frac{\pi}{\sqrt{2}}} \cdot \frac{1 + \frac{1.3}{4^2}(1-x^2) + \frac{1.3.5.7}{4^2.8^2}(1-x^2)^2 + \&c}{1 + \frac{1.3}{4^2}x^2 + \frac{1.3.5.7}{4^2.8^2}x^4 + \&c}$$

$$= F\left(\frac{2x}{1+x}\right).$$

$$1 - \frac{f^5(-x, -x^4)}{f(-x^5, -x^{10})} = 5x\, d \log_6 \frac{f(-x^2, -x^3)}{f(-x, -x^4)} \Big/ dx$$

$$f(u_1, v_1) = f(u_n, v_n) + u_1 f\left(\frac{v_{n-1}}{u_1}, \frac{u_{n+1}}{u_1}\right) + v_1 f\left(\frac{u_{n-1}}{v_1}, \frac{v_{n+1}}{v_1}\right)$$

$$+ u_2 f\left(\frac{v_{n-2}}{u_2}, \frac{u_{n+2}}{u_2}\right) + v_2 f\left(\frac{u_{n-2}}{v_2}, \frac{v_{n+2}}{v_2}\right) + \&c$$

If $ab = cd$, then

$$f(a,b) f(c,d) f\left(an, \frac{b}{n}\right) f\left(cn, \frac{d}{n}\right)$$

$$- f(-a,-b) f(-c,-d) f\left(-an, -\frac{b}{n}\right) f\left(-cn, -\frac{d}{n}\right)$$

$$= 2a f\left(\frac{c}{a}, ad\right) f\left(\frac{d}{an}, acn\right) f\left(n, \frac{ab}{n}\right) \psi(ab)$$

(Deduction).

x. $f(p,q)\, f(q,kp) = f(p,q)\, f(k,k^3)$ where $k = pq$

xi. $f(p,q) + f(-p,-q) = 2 f(p^3 q, pq^3)$

xii. $f(p,q) - f(-p,-q) = 2p\, f\left(\frac{q}{p}, \frac{p}{q} k^4\right)$ where $k = pq$

xiii. $f(p,q)\, f(-p,-q) = f(-p^2,-q^2)\, f(-pq,-pq)$

xiv. $f^2(p,q) + f^2(-p,-q) = 2 f(p^2,q^2)\, f(pq,pq)$

xv. $f^2(p,q) - f^2(-p,-q) = 4p\, f\left(\frac{q}{p}, \frac{p}{q} k^4\right) f(k^2,k^6)$ where $k = pq$

N.B. Of all the functional forms of $f(p,q)$ the most important are $f(-x,-x)$, $f(-x,-x^2)$ and $f(-x,-x^3)$; $f(x,x) = \phi(x)$ and $f(x,x^3) = \psi(x)$

Ex 1. $f(-x,x) = \phi(-x^4)$

2. $f(x,x^7)\, f(x^3,x^5) = \psi(x)\, \psi(x^4)$

3. $2f(x^3,x^5) = \psi(\sqrt{x}) + \psi(-\sqrt{x})$

4. $2f(x,x^7) = \dfrac{\psi(\sqrt{x}) - \psi(-\sqrt{x})}{\sqrt{x}}$

17 i. $\phi(x) = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \&c$

$= \frac{1+x}{1-x} \cdot \frac{1-x^2}{1+x^2} \cdot \frac{1+x^3}{1-x^3} \cdot \frac{1-x^4}{1+x^4}\ \&c$

ii. $\psi(x) = 1 + x + x^3 + x^6 + x^{10} + \&c$

$= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^3} \cdot \frac{1-x^6}{1-x^5}\ \&c$

iii. $\frac{1}{2} \log_e \phi(x) = \frac{x}{1+x} + \frac{x^3}{3(1+x^3)} + \frac{x^5}{5(1+x^5)} + \&c$

iv. $\log_e \psi(x) = \frac{x}{1+x} + \frac{x^2}{2(1+x^2)} + \frac{x^3}{3(1+x^3)} + \&c$

$\frac{\psi(x)}{\phi(x)} = \frac{1+x^2}{1+x} \cdot$ [illegible] $\&c$

$$\phi(x) + \phi(x^2) = \frac{1 \quad 2}{(1-x)(1-x^4)(1-x^7)(1-x^9)(1-x^{12})(1-x^{15})(1-x^{17})}$$

$$\phi(x) - \phi(x^2) =$$

If $F(x^2) = t$

then $$F\left(\frac{2x}{1+x}\right) \doteq \frac{\sqrt{t}}{1+(1-t)(1-t+\frac{3}{4}t^2)} \text{ nearly.}$$

$$\left(x + \frac{x^2}{2} + \frac{21x^3}{64} + \frac{31x^4}{128} + \&c\right)^{11}$$

$$= x^{11} + \frac{11}{2}x^{12} + \frac{1111}{64}x^{13} + \frac{11111}{896} + \frac{111111}{2688}x^{14} + \&c$$

θ is less than $\frac{1}{2160}\left\{\frac{\log x}{8 + .14(\log x)^2}\right\}^5$.

or $$F(1-e^{-x}) = \frac{x}{10+\sqrt{36+x^2}} - \frac{1}{2160}\left(\frac{x}{8+.14x^2}\right)^5.$$

$$\frac{1}{\phi(x^2)} = \frac{1}{\phi(x) \pm \phi(x^2)} + \frac{1}{\phi(-x) \pm \phi(x^2)}$$

$$\pm\frac{1}{\phi(-x^2)} = \frac{1}{\phi(x) \pm \phi(-x^2)} + \frac{1}{\phi(-x) \pm \phi(-x^2)}$$

vi. $\phi(x) + \phi(-x) = 2\phi(x^4)$

vii. $\phi(x) - \phi(-x) = 4x\,\psi(x^8)$

viii. $\phi(x)\,\phi(-x) = \phi^2(-x^2)$

ix. $\phi(x)\,\psi(x^2) = \psi^2(x)$

x. $\phi^2(x) - \phi^2(-x) = 8x\,\psi^2(x^4)$

xi. $\phi^2(x) + \phi^2(-x) = 2\phi^2(x^2)$

xii. $\phi^4(x) - \phi^4(-x) = 16x\,\psi^4(x^2)$

xiii. $\psi^2(x) + \psi^2(-x) = 2\,\psi(x^2)\,\phi(x^4)$

xiv. If $\left(\frac{1-z}{1+z}\right)^2 = \left\{\frac{\phi(-x)}{\phi(x)}\right\}^4$ then $1 - z^2 = \left\{\frac{\phi(-x^2)}{\phi(x^2)}\right\}^4$ [illegible]

Ex. 1. $\frac{\psi(x)}{\psi(-x)} = \sqrt{\frac{\phi(x)}{\phi(-x)}}$

2. $\psi(x)\,\psi(-x) = \psi(x^2)\,\phi(-x^2)$

3. $\frac{\psi(x)\,\psi(-x)}{\psi(x^2)\,\psi(-x^2)} = \frac{\psi(-x^2)}{\psi(x^4)}$

18. If $F(x) = e^{-\pi\cdot\frac{1 + \left(\frac{1}{2}\right)^2(1-x) + \left(\frac{1\cdot3}{2\cdot4}\right)^2(1-x)^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2(1-x)^3 + \&c}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 x^3 + \&c}}$

i. $F(x) = \frac{x}{16}\, e^{4\cdot\frac{\left(\frac{1}{2}\right)^2\frac{1}{1\cdot2}x + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac{1}{1\cdot2} + \frac{1}{3\cdot4}\right)x^2 + \&c}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c}}$

ii. $F\left(1 - \frac{1}{x}\right) + \theta = \frac{\log_e x}{10 + \sqrt{36 + (\log_e x)^2}}$ ~~where θ is numerically~~
~~much less than $\frac{2}{135}F^5\left(1 - \frac{1}{x}\right)$~~ $\theta = \frac{1}{2160}\left\{\frac{\log_e x}{8 + (\log_e x)^2}\right\}^5$.

iii. $\log_e F(x)\,\log_e F(1-x) = \pi^2$

iv. $F(\ldots) + F(1 - \ldots) = 1$ [illegible]

v. $F\left\{\frac{4x}{(1+x)^2}\right\} = \sqrt{F(x^4)}$

N.B. If we know the [illegible] of $F\left(\frac{2x}{1+x}\right)$ [illegible] we can [illegible] its expansion [illegible]

[illegible]

$$2\frac{\psi'(x)}{\psi(x)} - 2x\frac{\psi'(x^2)}{\psi(x^2)} = \frac{\phi'(x)}{\phi(x)}$$

$$\frac{\phi'(-x)}{\phi(-x)} - 4x\frac{\phi'(-x^2)}{\phi(-x^2)} = \frac{\phi'(x)}{\phi(x)}.$$

7. $F(x) = z$ & $y = \sqrt{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1 \cdot 3}{2 \cdot 4}\right)^2 x^2 + \&c}$, then

$\phi(z) = y$; $\phi(-z) = y\sqrt[4]{1-x}$.

$\phi(z^2) = y\sqrt{\dfrac{1+\sqrt{1-x}}{2}}$; $\phi(-z^2) = y\sqrt[8]{1-x}$.

$\phi(z^4) = y \cdot \dfrac{1+\sqrt[4]{1-x}}{2}$.

$\phi(\sqrt{z}) = y\sqrt{1+\sqrt{x}}$; $\phi(-\sqrt{z}) = y\sqrt{1-\sqrt{x}}$.

$\phi(\sqrt[4]{z}) = y(1+\sqrt[4]{x})$; $\phi(-\sqrt[4]{z}) = y(1-\sqrt[4]{x})$.

$\psi(z) = \dfrac{y}{\sqrt{2}}\sqrt[8]{\dfrac{x}{z}}$.

$\psi(z^2) = \dfrac{y}{2}\sqrt[4]{\dfrac{x}{z}}$.

$\psi(z^4) = \dfrac{y}{2}\sqrt{\dfrac{1-\sqrt{1-x}}{2z}}$.

$\psi(z^8) = y \cdot \dfrac{1-\sqrt[4]{1-x}}{4z}$.

$\psi(\sqrt{z}) = y\sqrt[4]{\dfrac{1+\sqrt{x}}{2}} \cdot \sqrt[16]{\dfrac{x}{z}}$.

$\psi(\sqrt[4]{z}) = y\sqrt{1+\sqrt[4]{x}} \cdot \sqrt[8]{\dfrac{1+\sqrt{x}}{2}} \cdot \sqrt[32]{\dfrac{x}{z}}$.

Writing $\frac{2x}{1+x^2}$ for x we have the expansion of $\varphi(x)$ to in terms i.e. that of $\left[\varphi\left\{\frac{4x}{(1+x)^2}\right\}\right]^2$ to in terms. Extracting the square root and expanding the result in ascending powers of $\frac{2x}{1+x^2}$ we can find the expansion of $\varphi\left(\frac{2x}{1+x^2}\right)$ to in terms.

vi. $\varphi\left(\frac{2x}{1+x}\right) = x + \frac{5}{16}x^3 + \frac{869}{2048}x^5 + \frac{4097}{32768}x^7 + \frac{15948955}{16777216}x^9 + \&c$

vii. $2F(1-e^{-8x}) = x - \frac{x^3}{3} + \frac{31}{120}x^5 - \frac{661}{2520}x^7 + \frac{219677}{725760}x^9 - \&c$

viii. $2F\left(1-e^{-\frac{8x}{1-x^2}}\right) = x + \frac{2}{3}x^3 + \frac{31}{120}x^5 + \frac{37}{7200}x^7 + \frac{5981}{725760}x^9 + \&c$

Ex 1. $\varphi(1) = ?$, $F(\frac{1}{2}) = e^{-\pi}$, $F(1) = 1$, $F\{(\sqrt{2}-1)^2\} = e^{-\pi\sqrt{2}}$

ix. $\varphi^2(x) = 1 + \left(\frac{1}{2}\right)^2\left\{1 - \frac{\varphi^4(-x)}{\varphi^4(x)}\right\} + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left\{1 - \frac{\varphi^4(-x)}{\varphi^4(x)}\right\}^2 + \&c$

x. $1 + \left(\frac{1}{2}\right)^2\left\{\frac{\varphi^4(-x)}{\varphi^4(x)}\right\} + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left\{\frac{\varphi^4(-x)}{\varphi^4(x)}\right\}^2 + \&c$

$= \frac{\varphi^2(x)}{n\varphi^2(x^n)}\left\{1 + \left(\frac{1}{2}\right)^2\left[\frac{\varphi^4(-x^n)}{\varphi^4(x^n)}\right] + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left[\frac{\varphi^4(-x^n)}{\varphi^4(x^n)}\right]^2 + \&c\right\}$

xi. $F\left\{\frac{\varphi^4(-x)}{\varphi^4(x)}\right\} = \sqrt[n]{F\left\{\frac{\varphi^4(-x^n)}{\varphi^4(x^n)}\right\}}$

xii. $F\left\{1 - \frac{\varphi^4(-x)}{\varphi^4(x)}\right\} = \sqrt[n]{F\left\{1 - \frac{\varphi^4(-x^n)}{\varphi^4(x^n)}\right\}}$

xiii. $F\left\{1 - \frac{\varphi^4(-x)}{\varphi^4(x)}\right\} = x$

xiv. $\varphi^2\sqrt{F(x)} = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c$

$$\begin{cases} 1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\&c = \dfrac{\sqrt[4]{\pi}}{\lfloor -\frac{1}{4}} = k \\ 1-2e^{-\pi}+2e^{-4\pi}-2e^{-9\pi}+\&c = \dfrac{k}{\sqrt[4]{2}} \end{cases}$$

$$\begin{cases} 1+2(e^{-\pi})^2+2(e^{-4\pi})^2+2(e^{-9\pi})^2+\&c = \dfrac{k}{2}\sqrt{2+\sqrt{2}} \\ 1-2(e^{-\pi})^2+2(e^{-4\pi})^2-2(e^{-9\pi})^2+\&c = \dfrac{k}{\sqrt[8]{2}} \end{cases}$$

$$\begin{cases} 1+2(e^{-\pi})^4+2(e^{-4\pi})^4+2(e^{-9\pi})^4+\&c = \dfrac{k}{2}\left(1+\dfrac{1}{\sqrt[4]{2}}\right) \\ 1-2(e^{-\pi})^4+2(e^{-4\pi})^4-2(e^{-9\pi})^4+\&c = \dfrac{k}{\sqrt{2}}\sqrt[4]{\sqrt[4]{2}+\sqrt[4]{8}} \end{cases}$$

$$\begin{cases} 1+2\sqrt{e^{-\pi}}+2\sqrt{e^{-4\pi}}+2\sqrt{e^{-9\pi}}+\&c = \dfrac{k}{\sqrt[8]{2}}\sqrt{1+\sqrt{2}} \\ 1-2\sqrt{e^{-\pi}}+2\sqrt{e^{-4\pi}}-2\sqrt{e^{-9\pi}}+\&c = \dfrac{k}{\sqrt[8]{2}}\sqrt{\sqrt{2}-1} \end{cases}$$

$$\begin{cases} 1+2\sqrt[4]{e^{-\pi}}+2\sqrt[4]{e^{-4\pi}}+2\sqrt[4]{e^{-9\pi}}+\&c = k\left(1+\dfrac{1}{\sqrt[4]{2}}\right) \\ 1-2\sqrt[4]{e^{-\pi}}+2\sqrt[4]{e^{-4\pi}}-2\sqrt[4]{e^{-9\pi}}+\&c = k\left(1-\dfrac{1}{\sqrt[4]{2}}\right) \end{cases}$$

$$\begin{cases} 1+e^{-\pi}+e^{-3\pi}+e^{-6\pi}+e^{-10\pi}+\&c = k\sqrt[8]{\dfrac{e^{\pi}}{32}} \\ 1+(e^{-\pi})^2+(e^{-3\pi})^2+(e^{-6\pi})^2+(e^{-10\pi})^2+\&c = k\sqrt[4]{\dfrac{e^{\pi}}{32}} \end{cases}$$

$$\begin{cases} 1+(e^{-\pi})^4+(e^{-3\pi})^4+(e^{-6\pi})^4+(e^{-10\pi})^4+\&c = \dfrac{k^2}{4}\sqrt{e^{\pi}(2-\sqrt{2})} \\ 1+(e^{-\pi})^8+(e^{-3\pi})^8+(e^{-6\pi})^8+\&c = \dfrac{k^2}{4}e^{\pi}\left(1-\dfrac{1}{\sqrt{2}}\right) \end{cases}$$

$$\begin{cases} 1+\sqrt{e^{-\pi}}+\sqrt{e^{-3\pi}}+\sqrt{e^{-6\pi}}+\&c = k\sqrt[4]{\dfrac{1+\sqrt[4]{2}}{2\sqrt{2}}}\sqrt[8]{e^{\pi}} \\ 1+\sqrt[4]{e^{-\pi}}+\sqrt[4]{e^{-3\pi}}+\sqrt[4]{e^{-6\pi}}+\&c = k\sqrt[32]{\dfrac{e^{\pi}}{2}}\sqrt[8]{1+\sqrt{2}}\sqrt{\dfrac{1+\sqrt[4]{2}}{\sqrt{8}}} \end{cases}$$

19. i.e. If $y = \pi \dfrac{1+(\frac{1}{2})^2(1-x)+(\frac{1.3}{2.4})^2(1-x)^2+\&c}{1+(\frac{1}{2})^2x+(\frac{1.3}{2.4})^2x^2+\&c}$, then

$$1+(\tfrac{1}{2})^2x+(\tfrac{1.3}{2.4})^2x^2+(\tfrac{1.3.5}{2.4.6})^2x^3+\&c$$
$$=\left(1+2e^{-y}+2e^{-4y}+2e^{-9y}+2e^{-16y}+\&c\right)^2$$

Cor. If $\sqrt{\alpha\beta}=\sqrt{\pi}$, then

$$\sqrt{\alpha}\left\{\tfrac{1}{2}+e^{-\alpha^2}+e^{-4\alpha^2}+e^{-9\alpha^2}+\&c\right\}$$
$$=\sqrt{\beta}\left\{\tfrac{1}{2}+e^{-\beta^2}+e^{-4\beta^2}+e^{-9\beta^2}+\&c\right\}$$

20. If $\alpha\beta=\pi$, then

$$\sqrt{\alpha}\left\{1+\frac{2}{(1+\alpha^2)^{n+1}}+\frac{2}{(1+4\alpha^2)^{n+1}}+\frac{2}{(1+9\alpha^2)^{n+1}}+\&c\right\}$$
$$=\frac{\lfloor n-\frac{1}{2}}{\lfloor n}\sqrt{\beta}\left\{1+2e^{-2\beta}\phi(2\beta)+2e^{-4\beta}\phi(4\beta)+\&c\right\}$$

where $\phi(x)=1+\frac{n}{n}\cdot\frac{x}{\lfloor 1}+\frac{n(n-1)}{n(n-\frac{1}{2})}\frac{x^2}{\lfloor 2}+\frac{n(n-1)(n-2)}{n(n-\frac{1}{2})(n-1)}\frac{x^3}{\lfloor 3}+\&c$

to $n+1$ terms.

$$=\frac{1}{\lfloor 2n}\left\{(2x)^n\lfloor n+\frac{n}{\lfloor 1}(2x)^{n-1}\lfloor n+1+\frac{n(n-1)}{\lfloor 2}(2x)^{n-2}\lfloor n+2\right.$$
$$\left.+\&c \text{ to infinity}\right\}$$

Cor. If $\alpha\beta=\pi$, then

$$\sqrt{\alpha}\left\{\tfrac{1}{2}+e^{-\alpha^2}+e^{-4\alpha^2}+e^{-9\alpha^2}+\&c\right\}$$
$$=\sqrt{\beta}\left\{\tfrac{1}{2}+e^{-\beta^2}+e^{-4\beta^2}+e^{-9\beta^2}+\&c\right\}$$

Ex. 1. $(1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\&c)^2$

$$=1+(\tfrac{1}{2})^2\tfrac{1}{2}+(\tfrac{1.3}{2.4})^2\tfrac{1}{2^2}+(\tfrac{1.3.5}{2.4.6})^2\tfrac{1}{2^3}+\&c=\frac{\sqrt{\pi}}{(\lfloor -\frac{1}{4})^2}$$

2. $\frac{1}{2}+e^{-\pi\sqrt{2}}+e^{-4\pi\sqrt{2}}+e^{-9\pi\sqrt{2}}+\&c=\frac{\lfloor [illegible]}{\sqrt{\pi}\lfloor [illegible]}$

3. $\frac{\pi-\frac{1}{2}}{\ldots}+\frac{4\pi-\ldots}{\ldots}+\frac{9\pi-\ldots}{\ldots}+\&c=\frac{1}{8}$

$$\begin{cases} (1-e^{-\pi})(1-e^{-2\pi})(1-e^{-3\pi})(1-e^{-4\pi}) \&c = k\,\dfrac{\sqrt[24]{e^{\pi}}}{\sqrt[8]{8}} \\ (1-e^{-2\pi})(1-e^{-4\pi})(1-e^{-6\pi})(1-e^{-8\pi}) \&c = \dfrac{k}{\sqrt{2}}\,\sqrt[12]{e^{\pi}}. \\ (1-e^{-4\pi})(1-e^{-8\pi})(1-e^{-12\pi}) \&c = \dfrac{k}{2}\,\sqrt[8]{2}\,\sqrt[6]{e^{\pi}}. \\ (1-e^{-8\pi})(1-e^{-16\pi})(1-e^{-24\pi}) \&c = \dfrac{k}{2}\,\sqrt[3]{e^{\pi}}\,\sqrt[4]{\tfrac{1}{2}\left(\sqrt[4]{2}-\tfrac{1}{\sqrt[4]{2}}\right)} \end{cases}$$

$$\begin{cases} (1-e^{-\pi})(1-e^{-3\pi})(1-e^{-5\pi}) \&c = \dfrac{\sqrt[8]{2}}{\sqrt[24]{e^{\pi}}} \\ (1-e^{-2\pi})(1-e^{-6\pi})(1-e^{-10\pi}) \&c = \dfrac{\sqrt[8]{8}}{\sqrt[12]{e^{\pi}}}. \\ (1-e^{-4\pi})(1-e^{-12\pi})(1-e^{-20\pi}) \&c = \dfrac{\sqrt[4]{2(2+\sqrt{2})\sqrt[4]{2}}}{\sqrt[6]{e^{\pi}}}. \end{cases}$$

$$\begin{cases} (1+e^{-\pi})(1+e^{-3\pi})(1+e^{-5\pi}) \&c = \dfrac{\sqrt[4]{2}}{\sqrt[24]{e^{\pi}}}. \\ (1+e^{-2\pi})(1+e^{-6\pi})(1+e^{-10\pi}) \&c = \dfrac{\sqrt[4]{(1+\sqrt{2})\sqrt[4]{2}}}{\sqrt[12]{e^{\pi}}}. \end{cases}$$

$$\left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(1+\frac{1}{3}\right)x^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(1+\frac{1}{3}+\frac{1}{5}\right)x^3 + \&c$$

$$= -\frac{1}{4}\left\{1+\left(\frac{1}{2}\right)^2 x + \&c\right\}\log(1-x).$$

Reconsider

$$\left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(1+\frac{1}{2}\right)x^2 + \&c$$

$$= -2\left\{1+\left(\frac{1}{2}\right)^2 x + \&c\right\}\Big\{\log\left[1+\left(\frac{1}{2}\right)^2 x + \&c\right]$$

$$+\frac{1}{3}\log(1-x) - \frac{1}{12}\log x - \frac{\pi}{12}\left[1+\left(\frac{1}{2}\right)^2(1-x)+\&c\right]\Big\}$$

$$= 4\left(\frac{1}{e^{x}-1} + \frac{1}{2}\cdot\frac{1}{e^{2x}-1} + \&c\right)$$

1. $h\phi(h) + h\phi(2h) + h\phi(3h) + h\phi(4h) + \&c$

$= \int_0^\infty \phi(x)\,dx + F(h)$ where $F(0)=0$.

$F(h)$ can be found by expanding the left side and writing the constant instead of a series.

Cor. If $h\phi(h) = ah^p + bh^q + ch^r + dh^s + \&c$ and if p, q, r, s &c be not negative, then

$h\phi(h) + h\phi(2h) + h\phi(3h) + h\phi(4h) + \&c$

$$= \int_0^\infty \phi(x)\,dx + a\frac{B_p}{p}h^p\cos\frac{\pi p}{2} + b\frac{B_q}{q}h^q\cos\frac{\pi q}{2} + \&c$$

N.B. $F(h)$ is not a terminating series; but if p, q, r, s &c be odd integers $F(h)$ appears to be 0 or to have some finite value. In this case $F(h)$ is really a function of $e^{-\frac{\pi}{h}}$ which is 0 when $h=0$ but which cannot be expressed as a function in ascending powers of h. When h is small the difference between the correct and the apparent value of $F(h)$ is so small that we can neglect it.

The apparent value of $x + \frac{2x}{1+x^2} + \frac{2x}{1+4x^2} + \&c$

$= \pi$, the real value being $\pi\left(1 + 2e^{-\frac{2\pi}{x}} + 2e^{-\frac{4\pi}{x}} + \&c\right)$

Ex. Show that $x\left\{\phi(0) + [\phi(x) + \phi(-x)] + [\phi(2x) + \phi(-2x)] + \&c\right\}$

$$= \int_{-\infty}^{\infty} \phi(x)\,dx + F\left(e^{-\frac{\pi}{x}}\right) \text{ where } F(0)=0.$$

2. $\frac{1^{m-1}}{e^{1^n x}} + \frac{2^{m-1}}{e^{2^n x}} + \frac{3^{m-1}}{e^{3^n x}} + \frac{4^{m-1}}{e^{4^n x}} + \&c$

$$= \frac{1}{n}\frac{\Gamma\left(\frac{m}{n}\right)}{x^{\frac{m}{n}}} + \frac{B_m}{m}\cos\frac{\pi m}{2} - \frac{x}{1!}\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2}$$

$$+ \frac{x^2}{2!}\frac{B_{m+2n}}{m+2n}\cos\frac{\pi(m+2n)}{2} - \&c$$

$$\frac{1^m . 1^n}{e^x-1} + \frac{2^m(2^n+1^n)}{e^{2x}-1} + \frac{3^m(3^n+1^n)}{e^{3x}-1} + \frac{4^m(4^n+2^n+1^n)}{e^{4x}-1} + \&c$$

$$= \frac{\lfloor m-n}{x^{m+1}} S_{m+1} S_{m-n+1} + \frac{\lfloor n}{x^{n+1}} S_{n+1} S_{n-m+1}$$

$$+ \frac{1}{x} S_{1-m} S_{1-n} - \frac{1}{2} S_{-m} S_{-n} + \frac{B_2}{\lfloor 2} . x\, S_{-1-m} S_{-1-n}$$

$$- \frac{B_4}{\lfloor 4} x^3 S_{-3-m} S_{-3-n} + \&c$$

$$1^m \{1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c\}$$
$$+ 2^m \{1^n e^{-2x} + 2^n e^{-4x} + 3^n e^{-6x} + \&c\}$$
$$+ 3^m \{1^n e^{-3x} + 2^n e^{-6x} + 3^n e^{-9x} + \&c\}$$
$$+ \quad \&c \quad \&c \quad \&c \quad \&c \quad \&c$$

$$= \frac{\lfloor m}{x^{m+1}} S_{1+m-n} + \frac{\lfloor n}{x^{n+1}} S_{1+n-m}$$

$$+ S_{-m} S_{-n} - \frac{x}{\lfloor 1} . S_{-m-1} S_{-n-1} + \frac{x^2}{\lfloor 2} S_{-m-2} S_{-n-2} \&c$$

N.B. ... is true only when n is positive.

3. $\frac{1^{l-1}}{(1+x^n)^m} + \frac{2^{l-1}}{(1+2^n x^n)^m} + \frac{3^{l-1}}{(1+3^n x^n)^m} + \&c$

$$= \frac{|\underline{\tfrac{l}{n}}\;|\underline{m-\tfrac{l}{n}-1}}{n x^l\;|\underline{m-1}} + \frac{B_l}{l}\cos\frac{\pi l}{2} - \frac{m}{|\underline{1}}\,x^n\,\frac{B_{l+n}}{l+n}\cos\frac{\pi(l+n)}{2}$$

$$+ \frac{m(m+1)}{|\underline{2}}\,x^{2n}\,\frac{B_{l+2n}}{l+2n}\cos\frac{\pi(l+2n)}{2} - \&c$$

4. $\frac{1^{m-1}}{e^{1^n x}-1} + \frac{2^{m-1}}{e^{2^n x}-1} + \frac{3^{m-1}}{e^{3^n x}-1} + \&c$

$$= \frac{1}{n}\,\frac{|\underline{\tfrac{m}{n}}}{x^{\frac{m}{n}}}\,S_{\frac{m}{n}} + \frac{1}{x}S_{1+n-m} - \frac{1}{2}\,\frac{B_m}{m}\cos\frac{\pi m}{2}$$

$$+ \frac{x}{|\underline{1}}\,\frac{B_2}{2}\,\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2} - \frac{x^3}{|\underline{3}}\,\frac{B_4}{4}\,\frac{B_{m+3n}}{m+3n}\cos\frac{\pi(m+3n)}{2}$$

$$+ \frac{x^5}{|\underline{5}}\,\frac{B_6}{6}\,\frac{B_{m+5n}}{m+5n}\cos\frac{\pi(m+5n)}{2} - \&c$$

Cor. $\frac{1^{n-1}}{e^{1^n x}-1} + \frac{2^{n-1}}{e^{2^n x}-1} + \frac{3^{n-1}}{e^{3^n x}-1} + \&c$

$$= \frac{C_0 - \frac{1}{n}\log_e x}{x} - \frac{1}{2}\,\frac{B_n}{n}\cos\frac{\pi n}{2} + \frac{x}{|\underline{1}}\,\frac{B_2}{2}\,\frac{B_{2n}}{2n}\cos\frac{2\pi n}{2}$$

$$- \frac{x^3}{|\underline{3}}\,\frac{B_4}{4}\,\frac{B_{4n}}{4n}\cos 2\pi n + \frac{x^5}{|\underline{5}}\,\frac{B_6}{6}\,\frac{B_{6n}}{6n}\cos 3\pi n - \&c$$

5. If $\phi(t) = \frac{1^{m-1}}{(e^{1^p x})^{t^q}} + \frac{2^{m-1}}{(e^{2^p x})^{t^q}} + \frac{3^{m-1}}{(e^{3^p x})^{t^q}} + \&c$

$$1^{n-1}\phi(1) + 2^{n-1}\phi(2) + 3^{n-1}\phi(3) + 4^{n-1}\phi(4) + \&c$$

$$= \left\{\frac{|\underline{\tfrac{m}{p}}}{m\,x^{\frac{m}{p}}}\,S_{1+\frac{m}{p}q-n}\right\} + \left\{\frac{|\underline{\tfrac{n}{q}}}{n\,x^{\frac{n}{q}}}\,S_{1+\frac{n}{q}p-m}\right\}$$

$$+ \frac{B_m}{m}\,\frac{B_n}{n}\cos\frac{\pi m}{2}\cos\frac{\pi n}{2} - \frac{x}{|\underline{1}}\,\frac{B_{m+p}}{m+p}\,\frac{B_{n+q}}{n+q}\cos\frac{\pi(m+p)}{2}\cos\frac{\pi(n+q)}{2}$$

$$+ \frac{x^2}{|\underline{2}}\,\frac{B_{m+2p}}{m+2p}\,\frac{B_{n+2q}}{n+2q}\cos\frac{\pi(m+2p)}{2}\cos\frac{\pi(n+2q)}{2} - \&c$$

$$\psi(x) - x\psi(x^9) = \frac{\phi(-x^9)}{\sqrt[3]{\phi(-x^3)}}\sqrt[3]{\psi(x^3)}.$$

$$\{3\phi(-x^9) - \phi(-x)\}^3 = 8\,\frac{\psi^3(x)}{\psi(x^3)}\,\phi(-x^3)$$

$$\frac{3617 + 16320\left(\frac{1^{15}x}{1-x} + \frac{2^{15}x^2}{1-x^2} + \frac{3^{15}x^3}{1-x^3} + \&c\right)}{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \&c\right)}$$
$$= 1617\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^3$$
$$+ 2000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^2$$

$$\frac{43867 - 28728\left(\frac{1^{17}x}{1-x} + \frac{2^{17}x^2}{1-x^2} + \frac{3^{17}x^3}{1-x^3} + \right)}{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \right)}$$
$$= 38367\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \right)\right\}^3$$
$$+ 5500\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \right)\right\}^2$$

$$\frac{174611 + 13200\left(\frac{1^{19}x}{1-x} + \frac{2^{19}x^2}{1-x^2} + \right)}{1 + 480\left(\frac{1^7x}{1-x} + \frac{2^7x^2}{1-x^2} + \right)}$$
$$= 53361\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \right)\right\}^3$$
$$+ 121250\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \right)\right\}^2$$

Cor. $\int_0^\infty \frac{x}{q} = \frac{\pi}{q} = h$, R.H.S becomes

$$\frac{1k}{m+n\,k}\left\{h\left(x\frac{1}{h-1} - C_0 - \log_e x\right) + C_1(m+n)\right\} + \frac{|m}{m}\cdot\frac{|n}{n}\cos\frac{\pi m}{2}\cos\frac{\pi n}{2}\ldots$$

6. $(1+2r\cos\theta+r^2)(1+2r^3\cos\theta+r^6)(1+2r^5\cos\theta+r^{10})$ &c

$\times(1-r^2)(1-r^4)(1-r^6)(1-r^8)$ &c.

$= 1 + 2r\cos\theta + 2r^4\cos 2\theta + 2r^9\cos 3\theta + 2r^{16}\cos 4\theta +$ &c

Cor. 1. $(1-r^3)(1-r^9)(1-r^{15})(1-r^{21})$ &c

$$= \frac{1 + r - (r^4 + 2r^9 + r^{16}) + (r^{25} + 2r^{36} + r^{49}) - \&c}{1 + r + r^3 + r^6 + r^{10} + r^{15} + r^{21} + \&c}$$

$$= \frac{1 + r\omega - (r^4\omega + 2r^9 + r^{16}\omega) + (r^{25}\omega + 2r^{36} + r^{49}\omega) - \&c}{1 + r\omega + r^3 + r^6 + r^{10}\omega + r^{15} + r^{21} + r^{28}\omega + \&c}$$

$$= \frac{1 - r^3 - r^{15} + r^{24} + r^{48} - \&c}{1 + r^9 + r^{27} + r^{54} + \&c}$$

$$= \frac{1 - 2r^9 + 2r^{36} - 2r^{81} + \&c}{1 + r^3 + r^6 + r^{15} + r^{21} + \&c}$$

2. If $(1+2r\sqrt{x}+r^2)(1+r^3\sqrt{x}+r^6)(1+r^5\sqrt{x}+r^{10})$ &c

$\times(1-r^2)(1-r^4)(1-r^6)(1-r^8)(1-r^{10})$ &c $= \phi(x)$

Find $\phi(1)$, $\phi(2)$, $\phi(3)$ and $\phi(4)$ in ascending powers of r.

7. If $\alpha\beta = \pi^2$ then

$$\sqrt[4]{\alpha}\, e^{-\frac{\alpha}{3}}(1-e^{-2\alpha})(1-e^{-4\alpha})(1-e^{-6\alpha})\ \&c$$

$$= \sqrt[4]{\beta}\, e^{-\frac{\beta}{3}}(1-e^{-2\beta})(1-e^{-4\beta})(1-e^{-6\beta})\ \&c$$

8. If $\alpha\beta = \pi^2$

$\frac{3}{\alpha}$ [illegible] $+ \frac{1}{(e^{2\alpha}-1)}$ [illegible] $\frac{2}{(e^{4\alpha}-1)}$ [illegible] $\frac{3}{(e^{6\alpha}-1)}$

$=$ [illegible] $\frac{1}{(e^{2\beta}-1)}$ [illegible]

$$1+480\left(\frac{1^7 x}{1-x}+\frac{2^7 x^2}{1-x^2}+\frac{3^7 x^3}{1-x^3}+\&c\right)$$

$$=\left\{1+240\left(\frac{1^3 x}{1-x}+\frac{2^3 x^2}{1-x^2}+\frac{3^3 x^3}{1-x^3}+\&c\right)\right\}^2.$$

$$\left\{1+240\left(\frac{1^3 x}{1-x}+\frac{2^3 x^2}{1-x^2}+\frac{3^3 x^3}{1-x^3}+\&c\right)\right\}$$

$$\times\left\{1-504\left(\frac{1^5 x}{1-x}+\frac{2^5 x^2}{1-x^2}+\frac{3^5 x^3}{1-x^3}+\&c\right)\right\}$$

$$=1-264\left(\frac{1^9 x}{1-x}+\frac{2^9 x^2}{1-x^2}+\frac{3^9 x^3}{1-x^3}+\&c\right).$$

$$\left\{1+240\left(\frac{1^3 x}{1-x}+\frac{2^3 x^2}{1-x^2}+\frac{3^3 x^3}{1-x^3}+\&c\right)\right\}$$

$$\times\left\{1-264\left(\frac{1^9 x}{1-x}+\frac{2^9 x^2}{1-x^2}+\frac{3^9 x^3}{1-x^3}+\&c\right)\right\}$$

$$=1-24\left(\frac{1^{13} x}{1-x}+\frac{2^{13} x^2}{1-x^2}+\frac{3^{13} x^3}{1-x^3}+\&c\right)$$

$$691+240.273\left(\frac{1^{11} x}{1-x}+\frac{2^{11} x^2}{1-x^2}+\frac{3^{11} x^3}{1-x^3}+\&c\right)$$

$$=441\left\{1+240\left(\frac{1^3 x}{1-x}+\frac{2^3 x^2}{1-x^2}+\frac{3^3 x^3}{1-x^3}+\&c\right)\right\}^3$$

$$+250\left\{1-504\left(\frac{1^5 x}{1-x}+\frac{2^5 x^2}{1-x^2}+\frac{3^5 x^3}{1-x^3}+\&c\right)\right\}^2.$$

If $S_{2n}=\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}x}{1-x}+\frac{2^{2n-1}x^2}{1-x^2}+\&c$ then it is possible to find S_{2n} in a series containing $S_4 S_{2n-4}$, $S_6 S_{2n-6}$ &c and not containing $S_2 S_{2n-2}$.

9. If $\alpha\beta = \pi^2$

$$\frac{1}{2} + \frac{2\alpha}{e^{2\alpha}-1} + \frac{4\alpha}{e^{4\alpha}-1} + \frac{6\alpha}{e^{6\alpha}-1} + \frac{8\alpha}{e^{8\alpha}-1} + \&c$$

$$+ \frac{2\beta}{e^{2\beta}-1} + \frac{4\beta}{e^{4\beta}-1} + \frac{6\beta}{e^{6\beta}-1} + \frac{8\beta}{e^{8\beta}-1} + \&c$$

$$= \frac{\alpha+\beta}{12}$$

$$\therefore \frac{2\pi}{e^{2\pi}-1} + \frac{4\pi}{e^{4\pi}-1} + \frac{6\pi}{e^{6\pi}-1} + \&c = \frac{\pi}{12} - \frac{1}{4}$$

10. If $\alpha\beta = \pi$

$$e^{\frac{n^2}{4}}\sqrt{\alpha}\left\{\frac{1}{2} + e^{-\alpha^2}\cos n\alpha + e^{-4\alpha^2}\cos 2n\alpha + e^{-9\alpha^2}\cos 3n\alpha + \&c\right\}$$

$$= \sqrt{\beta}\left\{\frac{1}{2} + e^{-\beta^2}\cosh n\beta + e^{-4\beta^2}\cosh 2n\beta + e^{-9\beta^2}\cosh 3n\beta + \&c\right\}$$

11. If $\alpha\beta = \pi$

$$e^{\frac{n^2}{[illegible]}} \cdot \frac{\sin n\alpha\,(1-2e^{-2\alpha^2}\cos 2n\alpha + e^{-4\alpha^2})(1-2e^{-4\alpha^2}[illegible])\ldots}{\sinh n\beta\,(1-2e^{-2\beta^2}\cosh 2n\beta + e^{-4\beta^2})(1-2e^{-4\beta^2}\cosh[illegible])\ldots}$$

$$= e^{\frac{\alpha^2-\beta^2}{6}}$$

12. If $\alpha\beta = \pi$

$$\left\{\frac{\alpha^2}{12} + \frac{\cos 2n\alpha}{1(e^{2\alpha^2}-1)} + \frac{\cos 4n\alpha}{2(e^{4\alpha^2}-1)} + \frac{\cos 6n\alpha}{3(e^{6\alpha^2}-1)} + \&c\right\}$$

$$-\left\{\frac{\beta^2}{12} + \frac{\cosh 2n\beta}{1(e^{2\beta^2}-1)} + \frac{\cosh 4n\beta}{2(e^{4\beta^2}-1)} + \frac{\cosh 6n\beta}{3(e^{6\beta^2}-1)} + \&c\right\}$$

$$= \frac{n^2}{2} + \frac{1}{2}\log_e \frac{\sin n\alpha}{\sinh n\beta}$$

13. $\alpha\beta = \pi$

$$\frac{n}{2} + \frac{\beta\sinh 2n\alpha}{[illegible]} + \frac{\alpha\sinh[illegible]}{[illegible]}$$

$$\frac{1^2 x}{(1-x)^2} + \frac{2^2 x^2}{(1-x^2)^2} + \frac{3^2 x^3}{(1-x^3)^2} + \frac{4^2 x^4}{(1-x^4)^2} + \&c$$

$$= \frac{1}{288}\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{288}\left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}^2.$$

$$\frac{1^4 x}{(1-x)^2} + \frac{2^4 x^2}{(1-x^2)^2} + \frac{3^4 x^3}{(1-x^3)^2} + \frac{4^4 x^4}{(1-x^4)^2} + \&c.$$

$$= \frac{1}{720}\left\{1 + 240\left(\frac{x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times\left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{720}\left\{1 - 504\left(\frac{x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \&c\right)\right\}.$$

$$\frac{1^6 x}{(1-x)^2} + \frac{2^6 x^2}{(1-x^2)^2} + \frac{3^6 x^3}{(1-x^3)^2} + \frac{4^6 x^4}{(1-x^4)^2} +$$

$$= \frac{1}{1008}\left\{1 + 480\left(\frac{x}{1-x} + \frac{2^7 x^2}{1-x^2} + \frac{3^7 x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{1008}\left\{1 - 504\left(\frac{x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times\left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$\frac{1^8 x}{(1-x)^2} + \frac{2^8 x^2}{(1-x^2)^2} + \&c$$

$$= \frac{1}{720}\left\{1 + 480\left(\frac{x}{1-x} + \frac{2^7 x^2}{1-x^2} + \frac{3^7 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times\left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{720}\left\{1 - 264\left(\frac{x}{1-x} + \frac{2^9 x^2}{1-x^2} + \frac{3^9 x^3}{1-x^3} + \&c\right)\right\},$$

$$+\frac{\beta^2 \sin 2n\beta}{e^{2\beta^2}-1}+\frac{\beta \sin 4n\beta}{e^{4\beta^2}-1}+\frac{\beta \sin 6n\beta}{e^{6\beta^2}-1}+\&c.$$

$$=\frac{\alpha \coth n\alpha-\beta \cot n\beta}{4}$$

14. If $\alpha\beta=\pi^2$ and n is a positive integer >1

$$\alpha^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\alpha}-1}+\frac{2^{2n-1}}{e^{4\alpha}-1}+\frac{3^{2n-1}}{e^{6\alpha}-1}+\&c\right\}$$

$$=(-\beta)^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\beta}-1}+\frac{2^{2n-1}}{e^{4\beta}-1}+\frac{3^{2n-1}}{e^{6\beta}-1}+\&c\right\}.$$

Cor 1. $\frac{1^5}{e^{2\pi}-1}+\frac{2^5}{e^{4\pi}-1}+\frac{3^5}{e^{6\pi}-1}+\frac{4^5}{e^{8\pi}-1}+\&c=\frac{1}{504}$

2. $\frac{1^9}{e^{2\pi}-1}+\frac{2^9}{e^{4\pi}-1}+\frac{3^9}{e^{6\pi}-1}+\frac{4^9}{e^{8\pi}-1}+\&c=\frac{1}{264}$

3. $\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\frac{4^{13}}{e^{8\pi}-1}+\&c=\frac{1}{24}$

15. If $\alpha\beta=\pi^2$ and n any integer

$$(4\alpha)^{1-n}\left\{\tfrac{1}{2}S_{2n-1}+\frac{1}{1^{2n-1}(e^{2\alpha}-1)}+\frac{1}{2^{2n-1}(e^{4\alpha}-1)}+\&c\right\}$$

$$-(-4\beta)^{1-n}\left\{\tfrac{1}{2}S_{2n-1}+\frac{1}{1^{2n-1}(e^{2\beta}-1)}+\frac{1}{2^{2n-1}(e^{4\beta}-1)}+\&c\right\}$$

$$=\frac{B_{2n}}{\lfloor 2n}\left\{(-\alpha)^n+\beta^n\right\}+\pi^2\frac{B_2}{\lfloor 2}\cdot\frac{B_{2n-2}}{\lfloor 2n-2}\left\{(-\alpha)^{n-1}+\beta^{n-1}\right\}$$

$$-\pi^4\frac{B_4}{\lfloor 4}\frac{B_{2n-4}}{\lfloor 2n-4}\left\{(-\alpha)^{n-2}+\beta^{n-2}\right\}+\&c$$ the last term

being $-\pi^n\frac{B_n}{\lfloor n}\frac{B_n}{\lfloor n}(-1)^{\frac{n}{2}}$ or $\pi^{n-1}\frac{B_{n-1}}{\lfloor n-1}\frac{B_{n+1}}{\lfloor n+1}$ [illegible]

according as n is even or odd.

If $P_n = \frac{B_n}{2n}(2^n-1)\cos\frac{\pi n}{2} + \frac{1^{n-1}x}{1+x} - \frac{2^{n-1}x^2}{1+x^2} + \frac{3^{n-1}x^3}{1+x^3} - \&c$

$$\& \; Q_n = \frac{\frac{1}{4}E_{n+1}\cos\frac{\pi n}{2} + \frac{1^n x}{1-x} - \frac{3^n x^3}{1-x^3} + \frac{5^n x^5}{1-x^5} - \&c}{\frac{1}{4}E_1 + \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \&c}$$

then $\frac{1}{2}Q_n = 2^n P_n - \frac{(n-1)(n-2)}{\underline{|2}} 2^{n-2} P_{n-2} Q_2$

$+ \frac{(n-1)(n-2)(n-3)(n-4)}{\underline{|4}} 2^{n-4} P_{n-4} Q_4 - \&c$

If $P_n = \frac{B_n}{2n}\cos\frac{\pi n}{2} + \frac{1^{n-1}x}{1-x} + \frac{2^{n-1}x^2}{1-x^2} + \frac{3^{n-1}x^3}{1-x^3} + \&c$

$$\& \; Q_n = \frac{1}{n+1} \cdot \frac{1^{n+1} - 3^{n+1}x + 5^{n+1}x^3 - 7^{n+1}x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$

then $\frac{1}{2}Q_n = -2^n P_n - \frac{(n-1)(n-2)}{\underline{|2}} 2^{n-2} P_{n-2} Q_2 -$

$\frac{(n-1)(n-2)(n-3)(n-4)}{\underline{|4}} 2^{n-4} P_{n-4} Q_4 - \&c$

$$\log_e \frac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\&c}{1 - 2x\cos\theta + 2x^4\cos 2\theta - 2x^9\cos 3\theta + \&c}$$

$$= 2\left\{\frac{x}{1-x^2}\cos\theta + \frac{x^2}{2(1-x^4)}\cos 2\theta + \frac{x^3}{3(1-x^6)}\cos 3\theta + \&c\right\}$$

Cor. 1. $\frac{1}{1^3(e^{2\pi}-1)} + \frac{1}{2^3(e^{4\pi}-1)} + \frac{1}{3^3(e^{6\pi}-1)} + \&c = \frac{7\pi^3}{360} - \frac{S_3}{2}$.

Cor. 2. $\frac{1}{1^7(e^{2\pi}-1)} + \frac{1}{2^7(e^{4\pi}-1)} + \frac{1}{3^7(e^{6\pi}-1)} + \&c = \frac{19\pi^7}{113400} - \frac{S_7}{2}$.

16. If $\alpha\beta = \pi$, $f(n) = \int_0^\infty e^{-x^2}\phi(nx)\,dx$ and $f(n)f(ni) = f(0)f(0)$, then

$$f(0)\sqrt{\alpha}\left\{\frac{1}{2}\phi(0) + e^{-\alpha^2}\phi(n\alpha) + e^{-4\alpha^2}\phi(2n\alpha) + e^{-9\alpha^2}\phi(3n\alpha) + \&c\right\}$$

$$= f(n)\sqrt{\beta}\left\{\frac{1}{2}\phi(0) + e^{-\beta^2}\phi(n\beta i) + e^{-4\beta^2}\phi(2n\beta i) + e^{-9\beta^2}\phi(3n\beta i) + \&c\right\}$$

17. $1 + \frac{2\cos nx}{1+x^2} + \frac{2\cos 2nx}{1+4x^2} + \frac{2\cos 3nx}{1+9x^2} + \&c$

$= \frac{\pi}{x}\coth\frac{\pi}{x}\cosh n - \frac{\pi}{x}\sinh n$

if n lies between 0 and $\frac{2\pi}{x}$.

18. $\phi(0) + \frac{\phi(xi)+\phi(-xi)}{1+x^2} + \frac{\phi(2xi)+\phi(-2xi)}{1+4x^2} + \&c$

$= \frac{\pi}{2x}\coth\frac{\pi}{x}\{\phi(1)+\phi(-1)\} + \frac{\pi}{2x}\{\phi(1)-\phi(-1)\}$.

N.B. Within certain limits of x only this theorem is true; so we must be very careful to apply the theorem. e.g.

$1 + \frac{2\cos mx\cos nx}{1+x^2} + \frac{2\cos 2mx\cos 2nx}{1+4x^2} + \&c$

$= \frac{\pi}{2x}\coth\frac{\pi}{x}\cosh m\cosh n - \sinh m\cosh n$

is true only when $m+n$ & $m-n$ lies between 0 and $\frac{2\pi}{x}$.

We may write $\phi\{f(0)+\psi(n)\}$ [illegible]

This theorem is always true [illegible]

$$\log_e \frac{1 + 4\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \&c\right)}{1 + 4\cos n\left(\frac{x\cos n}{1-x} - \frac{x^3\cos 3n}{1-x^3} + \frac{x^5\cos 5n}{1-x^5} - \&c\right)}$$

$$= 4\left\{\frac{x\sin^2 n}{1(1+x)} - \frac{x^2\sin^2 2n}{2(1+x^2)} + \frac{x^3\sin^2 3n}{3(1+x^3)} - \&c\right\}$$

$$\frac{1}{4}\log_e \frac{\sin n - x\sin 3n + x^3\sin 5n - x^6\sin 7n + \&c)}{\sin n(1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c)}$$

$$= \left\{\frac{x\sin^2 n}{1(1-x)} + \frac{x^2\sin^2 2n}{2(1-x^2)} + \frac{x^3\sin^2 3n}{3(1-x^3)} + \&c\right\}.$$

$$\frac{1}{4}\log_e \frac{\sin 2n - x\sin 4n + x^5\sin 8n - x^8\sin 10n + \&}{\sin 2n(1 - 2x + 4x^5 - 5x^8 + 7x^{16} - \&)}$$

$$= \frac{x}{1+x}\sin^2 n + \frac{x^2\sin^2 2n}{2(1+x^2)} + \frac{x^3\sin^2 3n}{3(1+x^3)} + \&c$$

$$+ \frac{x^4}{1-x^4}\sin^2 2n + \frac{x^8}{2(1-x^8)}\sin^2 4n + \&$$

$$\frac{1}{4}\log_e \frac{\sin n - x\sin 5n + x^2\sin 7n - x^5\sin 11n + \&}{\sin n(1 - 5x + 7x^2 - 11x^5 + 13x^7 - \&)}$$

$$= \frac{x\sin^2 n}{1-x} + \frac{x^2\sin^2 2n}{2(1-x^2)} + \frac{x^3\sin^2 3n}{3(1-x^3)} + \&$$

$$+ \frac{x\sin^2 2n}{1+x} + \frac{x^2\sin^2 4n}{2(1+x^2)} + \frac{x^3\sin^2 6n}{3(1+x^3)} + \&c.$$

19. If $\alpha\beta = \pi$ then

$$\left\{\frac{\alpha^2}{6}\phi(0) + \frac{\alpha x}{2}\phi'(0) + \frac{x^2}{4}\phi''(0)\right\}$$

$$+ \frac{1}{\alpha}\left\{\frac{\phi(n\alpha) + \phi(-n\alpha)}{1(e^{2\alpha^2}-1)} + \frac{\phi(2n\alpha) + \phi(-2n\alpha)}{2(e^{4\alpha^2}-1)} + \frac{\phi(3n\alpha) + \phi(-3n\alpha)}{3(e^{6\alpha^2}-1)} + \&c\right\}$$

$$+ \left\{\phi(n\alpha) + \frac{1}{2}\phi(2n\alpha) + \frac{1}{3}\phi(3n\alpha) + \&c\right\}$$

$$= \left\{\frac{\beta^2}{6}\phi(0) + \frac{\beta x i}{2}\phi'(0) + \frac{\alpha\beta i}{2}\phi(0)\right\}$$

$$+ \left\{\frac{\phi(n\beta i) + \phi(-n\beta i)}{1(e^{2\beta^2}-1)} + \frac{\phi(2n\beta i) + \phi(-2n\beta i)}{2(e^{4\beta^2}-1)} + \frac{\phi(3n\beta i) + \phi(-3n\beta i)}{3(e^{6\beta^2}-1)} + \&c\right\}$$

$$+ \left\{\phi(n\beta i) + \frac{1}{2}\phi(2n\beta i) + \frac{1}{3}\phi(3n\beta i) + \&c\right\}$$

If $\alpha\beta = \pi$, m any even integer and $\psi(m) = \frac{B_m}{\lfloor m}\{(\alpha i)^m + \beta^m\}$
$+ \frac{B_2}{\lfloor 2}\frac{B_{m-2}}{\lfloor m-2}(\alpha\beta)^2\{(\alpha i)^{m-4} + \beta^{m-4}\} - \frac{B_4}{\lfloor 4}\frac{B_{m-4}}{\lfloor m-4}(\alpha\beta)^4\{(\alpha i)^{m-8} + \beta^{m-8}\}$
$+ \&c$ the last term being $-(\alpha\beta i)^{\frac{m}{2}-1}\frac{B_{\frac{m}{2}-1}}{\lfloor \frac{m}{2}-1}\frac{B_{\frac{m}{2}+1}}{\lfloor \frac{m}{2}+1}(\beta i) + \ldots$
or $(\alpha\beta i)^{\frac{m}{2}}\frac{B_{\frac{m}{2}}}{\lfloor \frac{m}{2}}\frac{B_{\frac{m}{2}}}{\lfloor \frac{m}{2}}$ according as $\frac{m}{2}$ is odd or even, then

$$\left\{\frac{\phi(2n\alpha) + \phi(-2n\alpha)}{(2\alpha)^{m-1}(e^{2\alpha^2}-1)} + \frac{\phi(4n\alpha) + \phi(-4n\alpha)}{(4\alpha)^{m-1}(e^{4\alpha^2}-1)} + \&c\right\} + \alpha\left\{\frac{\phi(2n\alpha)}{(2\alpha)^{m-1}} + \frac{\phi(4n\alpha)}{(4\alpha)^{m-1}} \ldots\right\}$$

$$= \left\{\frac{\phi(2n\beta i) + \phi(-2n\beta i)}{(2\beta i)^{m-1}(e^{2\beta^2}-1)} + \frac{\phi(4n\beta i) + \phi(-4n\beta i)}{(4\beta i)^{m-1}(e^{4\beta^2}-1)} \ldots\right\}$$

$$+ \frac{\beta i}{2}\left\{\frac{\phi(2n\beta i) + \phi(-2n\beta i)}{(2\beta i)^{m-1}} + \frac{\phi(4n\beta i) + \phi(-4n\beta i)}{(4\beta i)^{m-1}} \ldots\right\}$$

$$+ \frac{\alpha}{2}\left\{\frac{B_{m-2}}{\lfloor m-2}\beta^{m-2}\frac{x}{\lfloor 2}\phi'(0) + \frac{B_{m-4}}{\lfloor m-4}\beta^{m-4}\frac{x^3}{\lfloor 3}\phi'''(0) + \ldots \frac{x^{m-1}}{\lfloor m-1}\phi^{m-1}\ldots\right\}$$

$$+ \phi(0)\,\psi(m) + \frac{x^2}{\lfloor 2}\phi''(0)\,\psi(m-2) + \ldots + \frac{x^m}{\lfloor m}\phi^m(0)$$

This theorem is always true whatever ...

$$\frac{\phi'(x)}{\phi(x)} - \frac{\psi'(x)}{\psi(x)} = \frac{1-\phi^4(-x)}{8x} \qquad \text{Show that}$$

$$\frac{\psi'(x)}{\psi(x)} - 2x\frac{\psi'(x^2)}{\psi(x^2)} = \frac{1-\phi^4(-x)}{8x} \qquad \frac{\phi'(x)}{\phi(x)} + \frac{\phi'(-x)}{\phi(-x)} = \frac{\phi^4(x) - \phi^4(-x)}{4x}$$

$$\frac{1^3 - 3^3x + 5^3x^3 - 7^3x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$

$$= 1 - 24\left\{\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right\}$$

$$4\,\frac{1^3 - 3^3x^4 + 5^3x^{12} - 7^3x^{24} + \&c}{1 - 3x^4 + 5x^{12} - 7x^{24} + \&c}$$

$$- \frac{1^3 - 3^3x + 5^3x^3 - 7^3x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c} = 3\phi^4(x)$$

$$\frac{1}{240}\cdot\frac{1^5 - 3^5x + 5^5x^3 - 7^5x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$

$$= \frac{1}{240} + \frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \&c$$

$$- 2\left\{\frac{x}{(1-x)^2} + \frac{4x^2}{(1-x^2)^2} + \frac{9x^3}{(1-x^3)^2} + \&c\right\}$$

1. $\left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \&c\right)$

$-\left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \&c\right)$

$= \frac{n}{x^2-1+} \frac{2^2-n^2}{1+} \frac{2^2}{x^2-1+} \frac{4^2-n^2}{1+} \frac{4^2}{x^2-1+} \&c$

2. $\phi(0) - \frac{1}{2}\phi(1) + \frac{1.3}{2.4}\phi(2) - \&c$

$= \phi(-\frac{1}{4}) - \frac{1}{2}\phi(-1\frac{1}{4}) + \frac{1.3}{2.4}\phi(-2\frac{1}{4}) - \&c$

3. $\frac{1}{4}\phi^2(x) = \frac{1}{4} + \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \&c$

4. $\frac{1}{8}\phi^4(x) = \frac{1}{8} + \frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1+x^4} + \&c$

5. If $\alpha\beta = \pi^2$, then

$\sqrt{\alpha}\left\{\frac{1}{4} + \frac{1}{e^{\alpha}-1} - \frac{1}{e^{3\alpha}-1} + \frac{1}{e^{5\alpha}-1} - \&c\right\}$

$= \sqrt{\beta}\left\{\frac{1}{4} + \frac{1}{e^{\beta}-1} - \frac{1}{e^{3\beta}-1} + \frac{1}{e^{5\beta}-1} - \&c\right\}$

6. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$ then

$\int_0^\infty \psi(x)\cos nx\,dx = \frac{\pi}{2}\phi(n).$

7. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then

(i) $\int_0^\infty \psi(x)\,dx = \frac{\pi}{2}\phi(0)$, (ii) $\int_0^\infty \psi^2(x)\,dx = \frac{\pi}{2}\int_0^\infty \phi^2(x)\,dx$

8. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then

(i) $\int_0^\infty x^r \phi(x)\cos nx \cos\frac{\pi r}{2}\,dx = \psi^r(n)$

(ii) $\int_0^\infty x^r \psi(x)\cos nx \cos\frac{\pi r}{2}\,dx = \frac{\pi}{2}\phi^r(n).$

$$1 - 5x + 7x^2 - 11x^5 + 13x^7 - \&c = \phi^2(-x)\, f(-x, -x^4) = B_x$$

$$f(-x, -x^2)\, f(-x^2, -x^9) = \phi(-x)\, \psi(x)$$

$$\frac{f(-x, -x^9)}{f(-x^4, -x^8)} = \frac{\phi(-x^2)}{\psi(x)}$$

$$1 - 2x + 4x^5 - 5x^8 + 7x^{16} - \&c = \psi(x^2)\, f^2(-x, -x^3) = A_x$$

Show that $\frac{1}{4}\phi^2(x)\,\psi^2(x^2) =$

$$\frac{1}{4} + \frac{x}{1-x} + \frac{x^2}{1+x^2} + \frac{7x^3}{1-x^3} + \frac{6x^4}{1+x^4} + \frac{5x^5}{1-x^5} + \frac{3x^6}{1+x^6}$$

$$\frac{1}{2}\phi(0) + \phi(1) + \phi(2) + \phi(3) + \&c$$

$$= \int_0^\infty \phi(x) - \frac{\phi(xi) - \phi(-xi)}{i(e^{2\pi x} - 1)}\, dx$$

$$A_x + 2x A_{x^4} = B_{x^8}$$

$$1 - 5(x^1) - 7(x^3)^2 + 11(x^3)^5 + 13(x^3)^7 - \&c$$

$$= \phi^2(x)\, \psi(-x) + 3x\,\phi^2(x^9).\psi(-x^9)$$

$$\frac{(1-x)^5(1-x^2)^5(1-x^3)^5(1-x^4)^5\,\&c}{(1-x^5)(1-x^{10})(1-x^{15})(1-x^{20})\,\&c}$$

$$= 1 - 5\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9} + \frac{11x^{11}}{1+x^{11}} - \frac{12x^{12}}{1+x^{12}}\right)$$

9. $\phi(x)\phi(x^2) = 1 + \frac{2x}{1-x} + \frac{2x^3}{1-x^3} - \frac{2x^5}{1-x^5} - \frac{2x^7}{1-x^7} + \&c$

10. $\phi(x)\phi(x^3) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} + \frac{2x^4}{1+x^4} - \frac{2x^5}{1-x^5} + \frac{2x^7}{1-x^7} - \&c$

shew it

11. $\phi(x)\phi(x^6) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} - \frac{2x^3}{1-x^3} + \frac{2x^5}{1-x^5} + \frac{2x^6}{1+x^6} - \frac{2x^7}{1-x^7} + \frac{2x^9}{1-x^9} - \frac{2x^{10}}{1+x^{10}}\ \&c$

12. $1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c$
$= \{(1-x)(1-x^2)(1-x^3)(1-x^4)\ \&c\}^3 = \phi^2(-x)\,\psi(x)$.

13. If $\alpha\beta = \frac{\pi}{4}$, then

$\sqrt{\alpha}\,\{\alpha e^{-\alpha^2} - 3\alpha e^{-9\alpha^2} + 5\alpha e^{-25\alpha^2}\ \&c\}$

$= \sqrt{\beta}\,\{\beta e^{-\beta^2} - 3\beta e^{-9\beta^2} + 5\beta e^{-25\beta^2}\ \&c\}$

14. If $\int_0^\infty \phi(x)\cos 2nx\,dx = \psi(n)$, then

$\alpha\{\tfrac{1}{2}\phi(0) + \phi(\alpha)\cos 2n\alpha + \phi(2\alpha)\cos 4n\alpha + \phi(3\alpha)\cos 6n\alpha + \&c\}$

$= \psi(n) + \psi(\beta-n) + \psi(\beta+n) + \psi(2\beta-n) + \psi(2\beta+n) + \&c$

with $\alpha\beta = \pi$ & n lying between 0 & β.

N.B. $\int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cos mx\,dx = \frac{\sin a}{e^m + 2\cos a + e^{-m}}$

$\int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \sin mx\,dx = \frac{1}{2}\,\frac{e^m - e^{-m}}{e^m + 2\cos a + e^{-m}}$

$\int_0^\infty \frac{\sin mx}{e^{2\pi x} - 1}\,dx = \frac{1}{2}\left(\frac{1}{e^m - 1} + \frac{1}{2} - \frac{1}{m}\right)$. $\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x}$ [illegible]

~~$\frac{1}{4}\frac{a}{m}$~~ [illegible]

$$y = \pi \frac{1 + \left(\frac{1}{2}\right)^2(1-x) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2(1-x)^2 + \cdots}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots}$$

then
$$\frac{dy}{dx} = -\frac{1}{x(1-x)\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^2}$$

$$\left(\frac{1}{2}\right)^2 + 2\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x + 3\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 x^2 + \cdots$$

$$= \left\{\frac{1}{3(1-x)} - \frac{1}{12x}\right\}\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}$$

$$+ \frac{1 - 24\left(\frac{1}{e^y - 1} + \frac{2}{e^{2y}-1} + \frac{3}{e^{3y}-1} + \cdots\right)}{12x(1-x)\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}}$$

$$1 + 240\left(\frac{1^3}{e^y - 1} + \frac{2^3}{e^{2y}-1} + \frac{3^3}{e^{3y}-1} + \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^4 (1 + 14x + x^2)$$

$$1 - 504\left(\frac{1^5}{e^y - 1} + \frac{2^5}{e^{2y}-1} + \frac{3^5}{e^{3y}-1} + \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^6 (1+x)(1 - 34x + x^2)$$

If x is changed to $\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2$ then y is changed to $2y$

$$1 + 240\left(\frac{1^3}{e^y - 1} + \frac{2^3}{e^{2y}-1} + \cdots\right) = \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^4 (1 - x + x^2)$$

$$1 - 504\left(\frac{1^5}{e^y - 1} + \frac{2^5}{e^{2y}-1} + \&c\right) = \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^6 (1+x)\left(1 - \tfrac{x}{2}\right)(1-2x).$$

$$1 - 8\left(\frac{1}{e^y + 1} - \frac{2}{e^{2y}+1} + \frac{3}{e^{3y}+1} - \&c\right) = \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \right\}^2 (1-x)$$

$$1 + 16\left(\frac{1^3}{e^y + 1} - \frac{2^3}{e^{2y}+1} + \frac{3^3}{e^{3y}+1} - \&c\right) = \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \right\}^4 (1-x^2)$$

$$1 - 8\left(\frac{1^5}{e^y + 1} - \frac{2^5}{e^{2y}+1} + \frac{3^5}{e^{3y}+1} - \&c\right) = \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \right\}^6 (1-x)(1-x+x^2)$$

$$17 + 32\left(\frac{1^7}{e^y + 1} - \frac{2^7}{e^{2y}+1} + \frac{3^7}{e^{3y}+1} - \&c\right) = \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^8 (1-x)(17 - 32x + x^2)$$

$$1 + 24\left(\frac{1}{e^y + 1} + \frac{2}{e^{2y}+1} + \frac{3}{e^{3y}+1} + \&c\right) = \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \right\}^2 (1+x)$$

$$1+4\left(\frac{1}{e^{y}+e^{-y}}+\frac{1}{e^{2y}+e^{-2y}}+\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2+\&c\right\}$$

$$4\left(\frac{1^2}{e^{y}+e^{-y}}+\frac{2^2}{e^{2y}+e^{-2y}}+\frac{3^2}{e^{3y}+e^{-3y}}+\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2+\&c\right\}^3\left(\frac{x}{4}\right)$$

$$4\left(\frac{1^4}{e^{y}+e^{-y}}+\frac{2^4}{e^{2y}+e^{-2y}}+\frac{3^4}{e^{3y}+e^{-3y}}+\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2+\&c\right\}^5\left\{\left(\frac{x}{4}\right)+\left(\frac{x}{4}\right)^2\right\}$$

$$4\left(\frac{1^6}{e^{y}+e^{-y}}+\frac{2^6}{e^{2y}+e^{-2y}}+\frac{3^6}{e^{3y}+e^{-3y}}+\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2+\&c\right\}^7\left\{\left(\frac{x}{4}\right)+11\left(\frac{x}{4}\right)^2+\left(\frac{x}{4}\right)^3\right\}$$

$$4\left(\frac{1^8}{e^{y}+e^{-y}}+\frac{2^8}{e^{2y}+e^{-2y}}+\frac{3^8}{e^{3y}+e^{-3y}}+\&c\right)$$

$$=\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2+\right\}^9\left\{\left(\frac{x}{4}\right)+57\left(\frac{x}{4}\right)^2+102\left(\frac{x}{4}\right)^3+\left(\frac{x}{4}\right)^4\right\}$$

If $\alpha\beta=\pi^2$ then

$$\alpha\left\{\frac{\sec n\alpha}{4}+\frac{\cos n\alpha}{e^{\alpha^2}-1}-\frac{\cos 3n\alpha}{e^{3\alpha^2}-1}+\frac{\cos 5n\alpha}{e^{5\alpha^2}-1}-\&c\right\}$$

$$=\beta\left\{\frac{1}{4}+\frac{\cosh 2n\beta}{e^{\beta^2}+e^{-\beta^2}}+\frac{\cosh 4n\beta}{e^{2\beta^2}+e^{-2\beta^2}}+\&c\right\}$$

$$1+4\left(\frac{1}{e^{y}-1}-\frac{1}{e^{3y}-1}+\frac{1}{e^{5y}-1}-\&c\right)$$

$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}.$$

$$1-4\left(\frac{1^2}{e^{y}-1}-\frac{3^2}{e^{3y}-1}+\frac{5^2}{e^{5y}-1}-\&c\right)$$

$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^3(1-x).$$

$$5+4\left(\frac{1^4}{e^{y}-1}-\frac{3^4}{e^{3y}-1}+\frac{5^4}{e^{5y}-1}-\&c\right)$$

$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^5(5-x)(1-x).$$

$$61-4\left(\frac{1^6}{e^{y}-1}-\frac{3^6}{e^{3y}-1}+\frac{5^6}{e^{5y}-1}-\&c\right)$$

$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^7(1-x)(61-46x+x^2)$$

$$1+240\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)$$

$$=\phi^8(-x)+256x\,\psi^8(x).$$

Similarly it is possible to express any identity in terms of $\phi(x)$ & $\psi(x)$.

$$\int\frac{\phi^{2n+4}(u)}{u}\,du=\int\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^n\frac{dx}{x(1-x)}$$

where $x=1-\dfrac{\phi^4(-u)}{\phi^4(u)}$

or $u=F(x)$ &c e^{-y}.

$$1-16\left(\frac{1^3}{e^{y}-1}-\frac{2^3}{e^{2y}-1}+\frac{3^3}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^4(1-x)^2.$$

$$1+8\left(\frac{1^5}{e^{y}-1}-\frac{2^5}{e^{2y}-1}+\frac{3^5}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^6(1-x)(1-x^2)$$

$$17-32\left(\frac{1^7}{e^{y}-1}-\frac{2^7}{e^{2y}-1}+\frac{3^7}{e^{3y}-1}-\&c\right).$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^8(1-x)^2(17-2x+17x^2)$$

$$31+8\left(\frac{1^9}{e^{y}-1}-\frac{2^9}{e^{2y}-1}+\frac{3^9}{e^{3y}-1}-\&c\right)$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^{10}(1-x)(1-x^2)(31-46x+31x^2).$$

$$\frac{1^3}{e^{y}-e^{-y}}+\frac{2^3}{e^{2y}-e^{-2y}}+\frac{3^3}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\right\}^4\frac{x}{16}$$

$$\frac{1^5}{e^{y}-e^{-y}}+\frac{2^5}{e^{2y}-e^{-2y}}+\frac{3^5}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^6\frac{x(1+x)}{16}$$

$$\frac{1^7}{e^{y}-e^{-y}}+\frac{2^7}{e^{2y}-e^{-2y}}+\frac{3^7}{e^{3y}-e^{-3y}}+\&c.$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^8\frac{x(1+6\tfrac{1}{2}x+x^2)}{16}.$$

$$\frac{1^9}{e^{y}-e^{-y}}+\frac{2^9}{e^{2y}-e^{-2y}}+\frac{3^9}{e^{3y}-e^{-3y}}+\&c$$
$$=\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^{10}\frac{x(1+x)(1+29x+x^2)}{16}.$$

$$\frac{1 + 24\left(\frac{1}{e^{y}+1} + \frac{2}{e^{2y}+1} + \frac{3}{e^{3y}+1} + \&c\right)}{\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^2 (1+x).}$$

$$1 - 240\left(\frac{1^3}{e^{y}+1} + \frac{2^3}{e^{2y}+1} + \frac{3^3}{e^{3y}+1} + \&c\right)$$

$$= \underbrace{\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}}_{z}{}^4 (1 - 16x + x^2).$$

$$1 + 504\left(\frac{1^5}{e^{y}+1} + \frac{2^5}{e^{2y}+1} + \frac{3^5}{e^{3y}+1} + \&c\right)$$

$$= z^6 (1+x)(1+29x+x^2).$$

$$1 + 24\left(\frac{1}{e^{2y}+1} + \frac{2}{e^{4y}+1} + \frac{3}{e^{6y}+1} + \&c\right)$$

$$= z^2\left(1 - \frac{x}{2}\right).$$

$$1 - 16\left(\frac{1^3}{e^{y}-1} - \frac{2^3}{e^{2y}-1} + \frac{3^3}{e^{3y}-1} - \&c\right)$$

$$= z^4 (1-x).$$

$$1 + 8\left(\frac{1^5}{e^{y}-1} - \frac{2^5}{e^{2y}-1} + \frac{3^5}{e^{3y}-1} - \&c\right)$$

$$= z^6 (1-x)\left(1 - \frac{x}{2}\right).$$

$$17 - 32\left(\frac{1^7}{e^{2y}-1} - \frac{2^7}{e^{4y}-1} + \frac{3^7}{e^{6y}-1} - \&c\right)$$

$$= z^8 (1-x)(17 - 17x + 2x^2)$$

$$1 + 240\left(\frac{1^3}{e^{4y}-1} + \frac{2^3}{e^{8y}-1} + \frac{3^3}{e^{12y}-1} + \&c\right) = z^4\left(1 - x + \frac{x^2}{16}\right)$$

$$1 - 504\left(\frac{1^5}{e^{4y}-1} + \frac{2^5}{e^{8y}-1} + \frac{3^5}{e^{12y}-1} + \&c\right) = z^6\left(1 - \frac{x}{2}\right)\left(1 - x \ldots \frac{\;}{32}\right)$$

If $\alpha\beta = \pi^2$, then

$$\frac{\sqrt{\alpha}}{\alpha^n}\left\{\frac{1}{2\alpha}\left(\frac{1}{1^{2n}}-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\&c\right)+\frac{1}{1^{2n}(e^{\alpha}-1)}-\frac{1}{3^{2n}(e^{3\alpha}-1)}+\frac{1}{5^{2n}(e^{5\alpha}-1)}-\frac{1}{7^{2n}(e^{7\alpha}-1)}+\&c\right\}$$

$$=\frac{\sqrt{\beta}}{\beta^n}\left\{(-1)^n\left[\frac{1}{2^{2n}(e^{\beta}+e^{-\beta})}+\frac{1}{4^{2n}(e^{2\beta}+e^{-2\beta})}+\&c\right]\right.$$

$$+\frac{1}{4}\left[\frac{\left(\frac{\beta}{2}\right)^{2n}}{\lfloor 2n}E_{2n+1}+\frac{B_2}{\lfloor 2}\cdot\frac{E_{2n-1}}{\lfloor 2n-2}\cdot\left(\frac{\beta}{2}\right)^{2n-1}(2\alpha)\right.$$

$$-\frac{B_4}{\lfloor 4}\cdot\frac{E_{2n-3}}{\lfloor 2n-4}\left(\frac{\beta}{2}\right)^{2n-2}(2\alpha)^2+\&c\ \&c$$

$$\left.\left.-\frac{(-\alpha\beta)^n}{\lfloor 2n}B_{2n}E_1\right]\right\}$$

$$\log_e\frac{1+\frac{4x}{1+x^2}+\frac{4x^2}{1+x^4}+\frac{4x^3}{1+x^6}+\&c}{1+\frac{4x\cos 2n}{1+x^2}+\frac{4x^2\cos 4n}{1+x^4}+\&c}$$

$$=8\left\{\frac{x\sin^2 n}{1(1-x^2)}+\frac{x^3\sin^2 3n}{3(1-x^6)}+\&c\right\}.$$

$$1+\frac{4x\cos n}{1+x^2}+\frac{4x^2\cos 2n}{1+x^4}+\frac{4x^3\cos 3n}{1+x^6}+\&c$$

$$\frac{\phi^3(-x^2)\{1-2x^2\cos 2n+2x^8\cos 4n-2x^{18}\cos 6n+\&c\}}{(1-2x\cos n+2x^4\cos 2n-2x^9\cos 3n+\&c)^2}$$

$$+ \frac{4x\cos n}{1+x^2} + \frac{4x^{[illegible]}\cos [illegible]}{1+x^{[illegible]}} + \frac{[illegible]}{1+x^{6}} + \ldots$$

$$\phi(x^2) \cdot \frac{1+2x\cos n + 2x^4\cos 2n + 2x^9\cos 3n + \&c}{1-2x\cos n + 2x^4\cos 2n - 2x^9\cos 3n + \&c}$$

$$\frac{1}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c = \frac{z^2}{4}\sqrt{x(1-x)}.$$

$$\frac{1^3}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^3}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^3}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c = \frac{z^4}{4}\sqrt{x(1-x)}\,(1-2x).$$

$$\frac{1^5}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^5}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^5}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c = \frac{z^6}{4}\sqrt{x(1-x)}\,(1-16x+16x^2).$$

$$\frac{1^7}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^7}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^7}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c = \frac{z^8}{4}\sqrt{x(1-x)}\,(1-2x)(1-136x+136x^2).$$

$$\frac{1^9}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^9}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^9}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c = \frac{z^{10}}{4}\sqrt{x(1-x)}\left\{1-1232x(1-x)+7936x^2(1-x)^2\right\}$$

of last term $= 2^{2n}(2^{2n}-1)\frac{B_{2n}}{2n}$.

If $\alpha\beta = \pi^2$ and n a positive integer then

$$\alpha^{n+\frac{1}{2}}\left\{\frac{1^{2n+1}}{e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\alpha}{2}}+e^{-\frac{5\alpha}{2}}}\right.$$

$$\left.\mp(-\beta)^{n+\frac{1}{2}}\left\{\frac{1^{2n+1}}{e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\beta}{2}}+e^{-\frac{5\beta}{2}}}\right.\right.$$

$$= 0$$

If $\alpha\beta = \pi$ then

$$\alpha\left\{\frac{\phi(\alpha)-\phi(-\alpha)}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{\phi(3\alpha)-\phi(-3\alpha)}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \&c\right\}$$

$$\mp \beta i\left\{\frac{\phi(\beta i)-\phi(-\beta i)}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{\phi(3\beta i)-\phi(-3\beta i)}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \&c\right\}$$

$$= 0$$

If $\alpha\beta = \pi$, then

$$\alpha\left\{\frac{\sin n\alpha}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{\sin 3n\alpha}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \&c\right\}$$

$$= \beta\left\{\frac{\sinh n\beta}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{\sinh 3n\beta}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \&c\right\}$$

If $\alpha\beta = \pi^2$ then

$$\left\{ \frac{1}{(e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}})} - \frac{1}{3(e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}})} + \frac{1}{5(e^{\frac{5\alpha}{2}}+e^{-\frac{5\alpha}{2}})} - \&c \right.$$

$$\left. + \frac{1}{(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}})} - \frac{1}{3(e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}})} + \frac{1}{5(e^{\frac{5\beta}{2}}+e^{-\frac{5\beta}{2}})} - \&c \right\}$$

$$= \frac{\pi}{8}.$$

If $\alpha\beta = \frac{\pi^2}{4}$ then

$$(\tan^{-1} e^{-\alpha} - \tan^{-1} e^{-3\alpha} + \tan^{-1} e^{-5\alpha} - \&c)$$
$$+ (\tan^{-1} e^{-\beta} - \tan^{-1} e^{-3\beta} + \tan^{-1} e^{-5\beta} - \&c) = \frac{\pi}{8}$$

$$\tan^{-1} e^{-\frac{y}{2}} - \tan^{-1} e^{-\frac{3y}{2}} + \tan^{-1} e^{-\frac{5y}{2}} - \&c$$

$$= \frac{1}{4} \sin^{-1} \sqrt{x}$$

$$\log_e(1-e^{-2y}) + \log_e(1-e^{-4y}) + \log_e(1-e^{-6y}) + \ldots$$

$$= \frac{1}{12}\left\{y + \log_e \frac{x(1-x)}{16}\right\} + \frac{1}{2}\log_e\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \ldots\right\}$$

$$1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)$$

$$= (1-2x)z^2 + 6x(1-x)z \cdot \frac{dz}{dx}$$

$$\frac{1}{e^{y}-e^{-y}}+\frac{3}{e^{3y}-e^{-3y}}+\frac{5}{e^{5y}-e^{-5y}}+\&c=\frac{x}{16}z^{2}$$

$$\frac{1^3}{e^{y}-e^{-y}}+\frac{3^3}{e^{3y}-e^{-3y}}+\frac{5^3}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^{4}\left(1-\frac{x}{2}\right)$$

$$\frac{1^5}{e^{y}-e^{-y}}+\frac{3^5}{e^{3y}-e^{-3y}}+\frac{5^5}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^{6}\left(1-x+x^{2}\right)$$

$$\frac{1^7}{e^{y}-e^{-y}}+\frac{3^7}{e^{3y}-e^{-3y}}+\frac{5^7}{e^{5y}-e^{-5y}}+\&c$$

$$=\frac{x}{16}z^{8}\left(1-\frac{x}{2}\right)\left(1-x+\frac{17}{2}x^{2}\right)$$

$$\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c$$

$$=x\frac{1+x}{(1-x)^2}-x^3\frac{1+x^2}{(1-x^2)^2}+x^6\frac{1+x^3}{(1-x^3)^2}-\&c$$

$$\frac{1+x}{1-x}-x\frac{1+x^3}{1-x^3}+x^3\frac{1+x^5}{1-x^5}-x^6\frac{1+x^7}{1-x^7}+\&c$$

$$=\psi(x)\,\phi(x^2)$$

$$\frac{1+x}{1-x}-x^2\frac{1+x^3}{1-x^3}+x^6\frac{1+x^5}{1-x^5}-x^{12}\frac{1+x^7}{1-x^7}+\&c$$

$$=\psi^2(x).$$

If $\alpha\beta = \pi^2$, then

$$\frac{\pi^2}{72} + \frac{1}{1^2(e^{\alpha}-e^{-\alpha})^2} + \frac{1}{2^2(e^{2\alpha}-e^{-2\alpha})^2} + \frac{1}{3^2(e^{3\alpha}-e^{-3\alpha})^2} + \&c$$

$$+ \frac{1}{1^2(e^{\beta}-e^{-\beta})^2} + \frac{1}{2^2(e^{2\beta}-e^{-2\beta})^2} + \frac{1}{3^2(e^{3\beta}-e^{-3\beta})^2} + \&c$$

$$-2\alpha\left\{1^2\log_e(1-e^{-2\alpha}) + 2^2\log_e(1-e^{-4\alpha}) + 3^2\log_e(1-e^{-6\alpha}) + \&c\right\}$$

$$-2\beta\left\{1^2\log_e(1-e^{-2\beta}) + 2^2\log_e(1-e^{-4\beta}) + 3^2\log_e(1-e^{-6\beta}) + \&c\right\}$$

$$= \frac{\alpha^2+\beta^2}{120}.$$

If $\alpha\beta = \pi^2$ and n any integer, then

$$\alpha^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}})} + \&c\right\}$$

$$+(-\beta)^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}})} + \&c\right\} =$$

$$= \frac{\pi}{2^{2n+1}}\left[\frac{E_1 E_{2n-1}}{\lfloor 2n-2}\left\{(-\alpha)^{n-1}+\beta^{n-1}\right\} - \frac{E_3 E_{2n-3}}{\lfloor 2 \lfloor 2n-4} \times \right.$$

$$\left\{(-\alpha)^{n-3}+\beta^{n-3}\right\} + \frac{E_5 E_{2n-5}}{\lfloor 4 \lfloor 2n-6}\left\{(-\alpha)^{n-5}+\beta^{n-5}\right\}$$

$- \&c$ the last term being $(-1)^{\frac{n-1}{2}}\left(\frac{E_n}{\lfloor n-1}\right)^2$ or

$(-1)^{\frac{n}{2}} \dfrac{E_{n-1}E_{n+1}}{\lfloor n-2 \lfloor n}(\alpha-\beta)$ according as n is odd or even.

$$\frac{1}{1^2(e^{\frac{y}{2}}+e^{-\frac{y}{2}})}+\frac{1}{3^2(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})}+\frac{1}{5^2(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})}+\&c$$

$$=\frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2x+\left(\frac{2\cdot4}{3\cdot5}\right)^2x^2+\left(\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\right)^2x^3+\&c}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^2+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2x^3+\&c}$$

$$\frac{1}{1^2(e^{\frac{y}{2}}-e^{-\frac{y}{2}})}-\frac{1}{3^2(e^{\frac{3y}{2}}-e^{-\frac{3y}{2}})}+\frac{1}{5^2(e^{\frac{5y}{2}}-e^{-\frac{5y}{2}})}-\&c$$

$$=\frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2\left\{1+\left(\frac{1}{2}\right)^2\right\}x+\left(\frac{2\cdot4}{3\cdot5}\right)^2\left\{1+\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2\right\}x^2+\&c}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^2+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2x^3+\&c}$$

$$\frac{1}{1^4(e^{\frac{y}{2}}+e^{-\frac{y}{2}})}+\frac{1}{3^4(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})}+\frac{1}{5^4(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})}+$$

$$=\frac{\sqrt{x}}{4}\cdot\frac{1+\left(\frac{2}{3}\right)^2\left[1+\left(\frac{2}{3}\right)^2\left\{1+\left(\frac{1}{2}\right)^2\right\}\right]x+\&c}{\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^2+\&c\right\}^3}$$

$$\sqrt{1-x}+\left(\frac{2}{3}\right)^2\sqrt{1-x}^{\,3}+\left(\frac{2\cdot4}{3\cdot5}\right)^2\sqrt{1-x}^{\,5}+$$

$$=\frac{\pi^2}{4}\left\{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-x)^2+\right\}$$

$$-2\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^2+\right\}$$

$$+4\left\{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1\cdot3}{2\cdot4}\right)^2x^2+\right\}\left\{\frac{1}{1^2(e^y+1)}-\frac{1}{3^2(e^{3y}+1)}\right.$$

$$\frac{1}{1(e^{\frac{y}{2}}-e^{-\frac{y}{2}})}+\frac{1}{3(e^{\frac{3y}{2}}-e^{-\frac{3y}{2}})}+\frac{1}{5(e^{\frac{5y}{2}}-e^{-\frac{5y}{2}})}+\&c$$

$$=\frac{\sqrt{x}}{4}\left(1+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+\&c\right)$$

$$\frac{1}{1(e^{\frac{y}{2}}+e^{-\frac{y}{2}})}-\frac{1}{3(e^{\frac{3y}{2}}+e^{-\frac{3y}{2}})}+\frac{1}{5(e^{\frac{5y}{2}}+e^{-\frac{5y}{2}})}-\&c$$

$$=\frac{\sqrt{x}}{4}\left\{1+\frac{1}{2}\cdot\frac{x}{3}+\frac{1.3}{2.4}\cdot\frac{x^2}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{7}+\&c\right\}.$$

If $P_1 e^{-y}+P_2 e^{-2y}+P_3 e^{-3y}+\&c$

$$=Q_1 x+Q_2 x^2+Q_3 x^3+\&c = f(x)$$

then $P_1 e^{-y}-P_2 e^{-2y}+P_3 e^{-3y}-\&c$

$$=Q_1\left(\frac{x}{1-x}\right)-Q_2\left(\frac{x}{1-x}\right)^2+Q_3\left(\frac{x}{1-x}\right)^3-\&c=-f\left(-\frac{x}{1-x}\right)$$

$$=f\left(-\frac{x}{1-x}\right).$$

If $F(e^{-y})=f(x)$ then $F(-e^{-y})=f\left(-\frac{x}{1-x}\right)$

$$1-\frac{4x}{1+x}+\frac{4x^3}{1+x^2}-\frac{4x^6}{1+x^3}+\frac{4x^{10}}{1+x^4}-\&c$$

$$=\left(1-2x+2x^4-2x^9+2x^{16}-\&c\right)^2$$

$$\frac{\psi^3(x)}{\psi(x^3)}=1+3\left(\frac{x}{1-x}-\frac{x^5}{1-x^5}+\frac{x^7}{1-x^7}-\frac{x^{11}}{1-x^{11}}+\&c\right)$$

$$\frac{\phi^3(x)}{\phi(x^3)}=1+6\left(\frac{x}{1-x}+\frac{x^2}{1+x^2}-\frac{x^4}{1+x^4}-\frac{x^5}{1-x^5}+\&c\right)$$

$$\left.\begin{aligned}\frac{\phi^3(x)}{\phi(x^3)}+2\,\frac{\phi^3(-x^2)}{\phi(-x^6)}&=3\,\phi(x)\,\phi(x^3)\\ \frac{\phi^2(x)}{\psi(x^3)}+\frac{\phi^3(-x^2)}{\phi(-x^6)}&=2\,\frac{\psi^3(x)}{\psi(x^3)}\end{aligned}\right\}$$

$$F\left(\frac{2-\sqrt3}{4}\right)=e^{-\pi\sqrt3}$$

3. $$\frac{1+\left(\frac12\right)^2(1-\alpha)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac12\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c}=3\,\frac{1+\left(\frac12\right)^2(1-\beta)+\&c}{1+\left(\frac12\right)^2\beta+\&c}$$

then $$\sqrt[8]{\frac{\alpha^3}{\beta}}-\sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}}=1.$$

and $$\sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}}-\alpha}\left\{1+\left(\frac12\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c\right\}$$

$$=\sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}}-\beta}\left\{1+\left(\frac12\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c\right\}$$

and $$\sqrt[8]{\frac{\alpha^3}{\beta}}+\sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}}=3\sqrt{\frac{\sqrt[8]{\frac{\alpha^3}{\beta}}-\alpha}{\sqrt[8]{\frac{\alpha^3}{\beta}}-\beta}}$$

If $\sin^3\theta = m^4 \sin\phi$ & $\cos^3\theta = n^4 \cos\phi$ and $m - n = 1$

then $\sqrt{\dfrac{m - \sin^2\theta}{m - \sin^2\phi}} = \dfrac{m+n}{3}$.

$$= \frac{1 + \left(\frac{1}{2}\right)^2 \sin^2\phi + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \sin^4\phi + }{1 + \left(\frac{1}{2}\right)^2 \sin^2\theta + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \sin^4\theta + }$$

If $\dfrac{\alpha^3}{\beta} = (1+p)^8$ & $\dfrac{(1-\alpha)^3}{1-\beta} = p^8$

then $\alpha = \dfrac{(1+p)^2(1-p^2)}{1+2p}$

& $\beta = \dfrac{(1-p)^2(1-p^2)}{(1+2p)^3}$.

$$1 - \alpha = p^3 \cdot \frac{p+2}{1+2p}.$$

$$1 - \beta = p \cdot \left(\frac{p+2}{1+2p}\right)^3.$$

$$\frac{1 + \left(\frac{1}{2}\right)^2 p \left(\frac{2+p}{1+2p}\right)^3 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 p^2 \left(\frac{2+p}{1+2p}\right)^6 + \&c}{(1+2p)\left\{1 + \left(\frac{1}{2}\right)^2 p^3 \cdot \frac{2+p}{1+2p} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 p^6 \left(\frac{2+p}{1+2p}\right)^2 \ldots\right\}}$$

If $p^3 \cdot \dfrac{2+p}{1+2p} = \dfrac{1 - \sqrt{1-x^3}}{2}$

then $p = \dfrac{-(1 + \sqrt{1-x}) + \sqrt{2 + x + 2\sqrt{1-x} \ldots}}{2}$

If $\sqrt[8]{\alpha} = m$ & $\sqrt[8]{\beta} = n$

$$m^4 + 2m^3n^3 - 2 \ldots n^4 = 0$$

$$F\left\{\frac{1}{2} - (2-\sqrt{3})\sqrt[4]{12}\right\} = e^{-3\pi}$$

$$\phi(e^{-3\pi}) = \frac{\phi(e^{-\pi})}{\sqrt[4]{6\sqrt{3}-9}}$$

~~$1 + \frac{n}{1}\cdot\frac{m}{2m}\cdot x\cdot\left(\frac{2+x}{1+2x}\right)^3 + \&c$~~

~~$= (1+2x)^{3n}\left\{1 + \frac{1}{2}\cdot\frac{n}{1}\, x^3\,\frac{2+x}{1+2x} + \&c\right\}$ if $m = \frac{2n+1}{6}$~~

$$x\,\frac{\psi^3(x^3)}{\psi(x)} = \frac{x}{1-x^2} - \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} - \frac{x^5}{1-x^{10}} + \frac{x^7}{1-x^{14}} - \frac{x^8}{1-x^{16}} + \frac{x^{10}}{1-x^{20}} - \&c$$

$$\frac{\phi^3(x^3)}{\phi(x)} = 1 - 2\left(\frac{x}{1+x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \frac{x^5}{1+x^5} + \frac{x^7}{1+x^7} - \frac{x^8}{1-x^8} + \frac{x^{10}}{1-x^{10}} - \frac{x^{11}}{1+x^{11}} + \&c\right)$$

$$\frac{\psi^3(-x^3)}{\psi(-x)} - \frac{\psi^3(x^3)}{\psi(x)} = 2x\,\frac{\psi^3(x^6)}{\psi(x^2)}$$

$$\frac{\psi^5(-x^5)}{\psi(-x)} - \frac{\psi^5(x^5)}{\psi(x)} = 4x^3\,\frac{\psi^5(x^{10})}{\psi(x^4)} + 2x\,\frac{f^5(-x^{20}, -x^{30})}{f(-x^4, -x^6)}$$

$$\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} = 1$$

$$\frac{\frac{1}{2} + e^{-\pi x}\cos(\pi\sqrt{1-x^2}) + e^{-4\pi x}\cos(4\pi\sqrt{1-x^2}) + \&c}{e^{-\pi x}\sin(\pi\sqrt{1-x^2}) + e^{-4\pi x}\sin(4\pi\sqrt{1-x^2}) + \&c}$$

$$= \frac{\frac{1}{2} + \sqrt{1+x}}{\sqrt{1-x}}$$

$$(\sqrt{5}+\sqrt{3})\left\{1 + 2e^{-\frac{\pi\sqrt{5}}{3}} + 2e^{-\frac{4\pi\sqrt{5}}{3}} + 2e^{-\frac{9\pi\sqrt{5}}{3}} + \&c\right\}$$

$$= (3+\sqrt{3})\left\{1 + 2e^{-3\pi\sqrt{5}} + 2e^{-12\pi\sqrt{5}} + 2e^{-27\pi\sqrt{5}} + \&c\right\}$$

$$\frac{1}{\phi^4(e^{-\pi})} = .71777$$

$$\phi(e^{-5\pi}) = \frac{\phi(e^{-\pi})}{\sqrt{5\sqrt{5}-10}}$$

$$\frac{\psi^3(x)}{\psi(x^3)} + \frac{\psi^3(-x)}{\psi(-x^3)} = 2\cdot\frac{\psi^3(x^2)}{\psi(x^6)}.$$

$$\frac{\psi^5(x)}{\psi(x^5)} + \frac{\psi^5(-x)}{\psi(-x^5)} + 2\,\frac{f^5(-x^4, -x^{16})}{f(-x^{20}, -x^{60})}$$

$$= 4\,\frac{\psi^5(x^2)}{\psi(x^{10})}.$$

$$\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}} - \sqrt[8]{\frac{\beta^5}{\alpha}} = 1 + \sqrt[3]{2}\cdot\sqrt[24]{\frac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}}$$

If 5. $\dfrac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c}$

$$= 5\,\frac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\beta)^2+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c}$$

then $\sqrt[8]{\dfrac{\alpha^5}{\beta}}-\sqrt[8]{\dfrac{(1-\alpha)^5}{1-\beta}}=1+\sqrt[3]{2}\cdot\sqrt[24]{\dfrac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$

If 9. $\left.\begin{array}{l}\dfrac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c}\\ =\dfrac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\beta)^2+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c}\end{array}\right\}=3\cdot\dfrac{1+\left(\frac{1}{2}\right)^2(1-\gamma)+}{1+\left(\frac{1}{2}\right)^2\gamma+}$

then $\sqrt[8]{\dfrac{\alpha}{\beta}}\sqrt[4]{\dfrac{1-\alpha}{1-\beta}}\left\{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c\right\}$

$\left\{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c\right\}$

$+3\left\{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c\right\}$

$=\left(\sqrt[8]{\dfrac{\alpha}{\beta}}+\sqrt[4]{\dfrac{1-\alpha}{1-\beta}}\right)\sqrt[4]{\left\{1+\left(\frac{1}{2}\right)^2\alpha+\&c\right\}\left\{1+\left(\frac{1}{2}\right)^2\beta+\&c\right\}}$

$$\frac{\phi^5(-x)}{\phi(-x^5)}+4\cdot\frac{(1-x)^5(1-x^2)^5(1-x^3)^5\cdots}{(1-x^5)(1-x^{10})(1-x^{15})\cdots}$$

$$=5\,\phi^3(-x)\,\phi(-x^5)$$

$$1+2\sqrt[8]{\frac{\beta^3}{\alpha}} \quad \& \quad 1+2\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}$$

$$\times 1+2\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} \quad \times 1+2\sqrt[3]{2}\sqrt[24]{\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$$

$$=3 \qquad\qquad =5$$

$$F\left\{\frac{1-\sqrt{1-(2-\sqrt{3})^4}}{2}\right\} = e^{-3\pi}$$

$$F\left\{\frac{1-\sqrt{1-(\sqrt{5}-2)^8}}{2}\right\} = e^{-5\pi}$$

$$F\left(\tfrac{1}{2}-\sqrt{\sqrt{5}-2}\right) = e^{-\pi\sqrt{5}}$$

$$F\left(\frac{8-3\sqrt{7}}{16}\right) = e^{-\pi\sqrt{7}}$$

$$1+\left(\tfrac{1}{2}\right)^2 p\left(\frac{2-p}{1+2p}\right)^5 + \left(\frac{1.3}{2.4}\right)^2 p^2\left(\frac{2-p}{1+2p}\right)^{10} + \&c$$

$$=(1+2p)\left\{1+\left(\tfrac{1}{2}\right)^2 p^5\frac{2-p}{1+2p} + \left(\frac{1.3}{2.4}\right)^2 p^{10}\left(\frac{2-p}{1+2p}\right)^2 + \&c\right\}$$

$$\phi(e^{-9\pi}) = \frac{1+\sqrt[3]{2(\sqrt{3}+1)}}{3}\,\phi(e^{-\pi})$$

$$1+\left(\tfrac{1}{2}\right)^2\frac{1-\frac{1-11p-p^2}{(1+2p)^2}\sqrt{\frac{1+p^2}{1+2p}}}{2} + \&c$$

$$=(1+2p)\left\{1+\left(\tfrac{1}{2}\right)^2\frac{1-(1+11p-p^2)\sqrt{\frac{1+p^2}{1+2p}}}{2} + \&c\right\}$$

$$\frac{\psi^7(x)}{\psi(x^7)} + \frac{\psi^7(-x)}{\psi(-x^7)} + 14 \cdot \frac{f^7(-x^8, -x^{16})}{f(-x^{56}, -x^{112})}$$

$$= 8\frac{\psi^7(x^4)}{\psi(x^{14})} + 2\frac{f^7(-x^2, -x^4)}{f(-x^{14}, -x^{28})} + 6\frac{f^7(-x^4, -x^8)}{f(-x^{28}, -x^{56})}$$

$$8\frac{\psi^7(x^2)}{\psi(x^{14})} + 2\frac{f^7(-x^2, -x^4)}{f(x^{14}, -x^{28})} - \frac{\psi^7(x)}{\psi(x^7)} - \frac{\psi^7(-x)}{\psi(-x^7)}$$

is a function of x^4

$$\frac{\left\{\phi^2(x) - \phi^2(x^7)\right\}^2 \frac{\phi^2(x^7)}{\phi(x)} - \left\{\phi^2(-x) - \phi^2(-x^7)\right\}^2 \frac{\phi^2(-x^7)}{\phi(-x)}}{x^3}$$

is function of x^4.

$$\phi(x^2) = \phi(x^2)\, e^{i\left\{\frac{\pi}{4} - \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}}$$

$$\phi^3(x^2)\,\phi^3(x^{14}) \sin\left\{3\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} - 3\tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$+ 2\phi(x^2)\,\phi^5(x^{14}) \sin\left\{5\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} - \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$+ \frac{\phi^7(x^{14})}{\phi(x^2)} \sin\left\{7\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} + \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$F\left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{6}+\sqrt{2}+1}\right) = e^{-\pi\sqrt{6}}$$

$$\frac{1-2\beta}{2\alpha-1} = \frac{f(x)}{f\left(\frac{\pi}{x}\right)}$$

$$F\left(\frac{(7-4\sqrt{3})(4-\sqrt{15})}{(\sqrt{5}+1)^4}\right) = e^{-\pi\sqrt{15}}$$

$$F\left(\frac{16-7\sqrt{3}\pm\sqrt{15}}{32}\right) = e^{-\pi\sqrt{15}}$$

$$F\left\{(2-\sqrt{3})^4(\sqrt{2}-1)^6\right\} = e^{-3\pi\sqrt{2}}$$

$$F\left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{2}-1}\right)^2 = e^{-\pi\sqrt{6}}$$

$$7.\ \frac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\cdots}{1+\left(\frac{1}{2}\right)^2\alpha+\cdots} = \frac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\cdots}{1+\left(\frac{1}{2}\right)^2\beta+\cdots}$$

VII $\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} = 1$

III $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1$

I. $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} = 1$

$$\sqrt{7\cdot\frac{\sqrt[8]{\alpha\beta}-\beta}{\alpha-\sqrt[8]{\alpha\beta}}}.$$

V $\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3}$

$= \sqrt[8]{\alpha^3\beta} + \sqrt[8]{(1-\alpha)^3(1-\beta)}.$

15c

III. $\sqrt[8]{\alpha\beta^5}+\sqrt[8]{(1-\alpha)(1-\beta)^5}$

$=\sqrt[8]{\alpha^5\beta}+\sqrt[8]{(1-\alpha)^5(1-\beta)}$

III $\frac{1-2\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{1-2\sqrt[4]{\alpha\beta}}=\sqrt{1+4\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}=\frac{1+(\text{[illegible]})^2\alpha+}{1+(\text{[illegible]})^2\beta+}$

VII $\frac{1-4\sqrt[24]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}}{1-2\sqrt[8]{\alpha\beta}}=\frac{1+(\text{[illegible]})^2\alpha+\text{[illegible]}}{1+(\text{[illegible]})^2\beta+\text{[illegible]}}$

V $\frac{1+\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}}}{1+\sqrt[8]{(1-\beta)^3(1-\beta)}}=\frac{1+(\text{[illegible]})^2\alpha+\text{[illegible]}}{1+(\text{[illegible]})^2\beta+}=\frac{1-\sqrt[8]{\frac{\beta}{\alpha}}}{1-\sqrt[8]{\alpha\text{[illegible]}}}$

III $\frac{-1+\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}}}{1-\sqrt[4]{(1-\alpha)(1-\beta)}}=\frac{1-\sqrt[4]{\frac{\beta^3}{\alpha}}}{1-\sqrt[4]{\alpha\beta}}=\frac{1+(\text{[illegible]})\alpha+}{\text{[illegible]}}$

VII $\sqrt{7\frac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}}-\sqrt[8]{\frac{\beta^7}{\alpha}}}{\sqrt[8]{\frac{\alpha^7}{\beta}}-\sqrt[8]{\frac{(1-\alpha)^7}{1-\beta}}}}=\frac{1+(\text{[illegible]})^2\alpha+}{1+(\text{[illegible]})^2\beta+}$

III $\sqrt{3\cdot\frac{1+\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{1+\sqrt[8]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}}}}=\frac{1+(\text{[illegible]})\alpha+\text{[illegible]}}{\text{[illegible]}}$

VII $\sqrt{7\cdot\frac{1-4\sqrt[24]{\text{[illegible]}}}{\text{[illegible]}}-1}$

157
146

III $\sqrt{3}\,\dfrac{1-2\sqrt[8]{\dfrac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{2\sqrt[8]{\dfrac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}}-1} = \dfrac{1+(\ldots)^{3}\alpha\ldots}{1+(\ldots)^{3}\beta\ldots}$

soluble from five [illegible]

XI $\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}+2\sqrt[3]{2}\,\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}=1$

XXIII $\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}+\sqrt[3]{4}\,\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}=1$

V $\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}+2\sqrt[3]{4}\,\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}=1$

II. $\sqrt{\dfrac{\beta}{\alpha}}+\sqrt{\dfrac{1-\beta}{1-\alpha}}-\sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\left(\dfrac{1+(\ldots)^{2}\alpha+\ldots}{1+(\ldots)^{2}\beta+\ldots}\right)^2$

I $\sqrt[4]{\dfrac{\beta}{\alpha}}+\sqrt[4]{\dfrac{1-\beta}{1-\alpha}}-\sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\dfrac{1+(\ldots)^{2}\alpha+\ldots}{1+(\ldots)^{2}\beta+\ldots}$

IX $\sqrt[8]{\dfrac{\beta}{\alpha}}+\sqrt[8]{\dfrac{1-\beta}{1-\alpha}}-\sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{\dfrac{1+(\ldots)\alpha+\ldots}{1+(\ldots)\beta+\ldots}}$

XV $\sqrt[8]{\dfrac{\beta}{\alpha}}+\sqrt[8]{\dfrac{1-\beta}{1-\alpha}}-\sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[12]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{\ldots}$

XIII $\sqrt[4]{\dfrac{\beta}{\alpha}}+\sqrt[4]{\dfrac{1-\beta}{1-\alpha}}-\sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[6]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{\dfrac{1+\ldots}{1+\ldots}}$

III $\sqrt{\dfrac{\beta}{\alpha}}+\sqrt{\dfrac{1-\beta}{1-\alpha}}-\sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}-8\sqrt[3]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\left\{\dfrac{1+(\ldots)\ldots}{\ldots}\right.$

XV $\sqrt[16]{\alpha\beta}\left(\sqrt[4]{1+\sqrt{\alpha}}\,\sqrt[4]{1+\sqrt{\beta}}+\sqrt[4]{1-\sqrt{\alpha}}\,\sqrt[4]{1-\sqrt{\beta}}\right)$

$+\sqrt[16]{(1-\alpha)(1-\beta)}\left(\sqrt[4]{1+\sqrt{1-\alpha}}\,\sqrt[4]{1+\sqrt{1-\beta}}+\sqrt[4]{1-\sqrt{1-\alpha}}\,\sqrt[4]{1-\sqrt{1-\beta}}\ldots\right.$ [illegible]

$=\sqrt{\ldots}$ [illegible]

$$\text{III} \quad \sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} + \sqrt[8]{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}} = 1$$

$$\text{V} \quad \sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3} + \sqrt[3]{2}\cdot\sqrt[24]{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}} = 1$$

$$\text{XXXI} \quad \sqrt[32]{\alpha\beta}\left\{\sqrt[8]{(1+\sqrt{\alpha})(1+\sqrt{\beta})}\sqrt{1+\sqrt[4]{\alpha\beta}+\sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})}} + \sqrt[8]{(1-\sqrt{\alpha})(1-\sqrt{\beta})}\sqrt{1+\sqrt[4]{\alpha\beta}+\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}}\right\}$$

$$+ \sqrt[32]{(1-\alpha)(1-\beta)}\left\{\text{&c \&c \&c}\right\} = \sqrt[4]{8}$$

$$F\left(\tfrac{1}{2} - 3\sqrt{5\sqrt{13}-18}\right) = e^{-\pi\sqrt{13}}$$

changing β to $\frac{4B}{(1+B)^2}$ & α to $1-B^2$ we get an equation in $\frac{4B(1-B)}{1+B}$ and the

value of B^2 is for $e^{-\pi\sqrt{2n}}$

If α I degree, β 3rd & γ 9th

then $1 + \sqrt[3]{4}\sqrt[24]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} = 3\sqrt{\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\alpha+\&c}}$

$$\text{III} \quad \sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$\text{V} \quad \sqrt[4]{\alpha(1-\beta)} + \sqrt[4]{\beta(1-\alpha)} = \sqrt[3]{4}\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}$$

Let $\alpha = \sin^2(u+v)$ & $\beta = \sin^2(u-v)$

III $\sin 2u = 2\sin v$; V $\sin 2u = \sin v(1+\cos v)$

VII $\sin 2u = 4\sin\frac{v}{2}\sqrt{\cos 2v + 3\cos v}$

$$F\left(\frac{1-\sqrt{1-(55\pm 12\sqrt{21})(8-3\sqrt{7})^2}}{2}\right) = e^{-\pi\sqrt{21}}$$ 160 154 149

$$F\left(\frac{1-\sqrt{1-\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{24}}}{2}\right) = e^{-7\pi}$$

$$F\left(\frac{1-\sqrt{1-(\sqrt{5}-2)^8(2-\sqrt{3})^8\left(\frac{\sqrt{4+\sqrt{15}}\pm\sqrt[4]{15}}{2}\right)^{24}}}{2}\right) = e^{-15\pi}$$

If α is 1st deg. β 3rd, γ 5th & δ 15th. then

$$\frac{\sqrt[8]{\beta\gamma}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{\alpha\delta}} = \frac{\sqrt[8]{(1-\beta)(1-\gamma)}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$$\phi(x)\phi(x^9) = \phi^2(x^3) + 2x\phi(-x^2)\psi(x^3)(1+x^3)(1+x^9)(1+x^{15})\&c$$

$$\frac{1+(\frac{1}{2})^2\beta+(\frac{1\cdot 3}{2\cdot 4})^2\beta^2+\&c}{\sqrt{1+(\frac{1}{2})^2\alpha+\&c}\,\sqrt{1+(\frac{1}{2})^2\gamma+\&c}} = 1-\sqrt[3]{4}\sqrt[24]{\frac{\gamma^3(1-\alpha)^3}{\beta(1-\beta)}} = \sqrt[3]{4}\sqrt[24]{\frac{\alpha^3(1-\gamma)^3}{\beta(1-\beta)}}-1$$

where α is 1st, β 3rd, & γ 9th deg.

$$\therefore \sqrt[8]{\alpha(1-\gamma)}+\sqrt[8]{\gamma(1-\alpha)} = \sqrt[3]{2}\cdot\sqrt[24]{\beta(1-\beta)}$$

$$\sqrt{\alpha\gamma}+\sqrt{(1-\alpha)(1-\gamma)}+2\sqrt[3]{4\beta(1-\beta)}$$
$$= 1+8\sqrt[4]{\beta(1-\beta)}\,\sqrt[8]{\alpha\gamma(1-\alpha)(1-\gamma)}$$

$$1+\sqrt[3]{4}\sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}} = \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$1-$ [illegible] $\sqrt[8]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}}\ \sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}}$ [illegible]

$$\psi(x) - 3x\,\psi(x^9) = \frac{\phi(-x)}{(1-x^3)(1-x^9)(1-x^{15})\,\&c}$$

$$\phi(x)\phi(x^{15}) - \phi(x^3)\phi(x^5)$$
$$= 2x\, f(-x^4) f(-x^{30})\,(1+x^3)(1+x^5)(1+x^{15})\,\&c$$
$$\times (1+x^5)(1+x^{15})(1+x^{25})\,\&c.$$

$$\phi(x)\phi(x^9) + \phi^2(x^3) = 2\psi(x)\phi(-x^{18})(1+x^3)(1+x^9)(1+\ldots$$

$$\phi(x)\phi(x^{15}) + \phi(x^3)\phi(x^5)$$
$$= 2 f(-x^6) f(-x^{10})\,(1+x)(1+x^3)(1+x^5)\,\&c$$
$$\times (1+x^{11})(1+x^{13})(1+x^{23})\,\&c$$

$$\psi(x^3)\psi(x^5) - x\,\psi(x)\psi(x^{15}) = \frac{f(-x)\, f(-x^{15})}{\chi(-x^3)\,\chi(-x^5)}$$

$$\psi(x^3)\psi(x^5) + x\,\psi(x)\psi(x^{15}) = \frac{f(-x^3)\, f(-x^5)}{\chi(-x)\,\chi(-x^{15})}$$

$$F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-13\pi}$$
when $x + \frac{1}{x} = \frac{\left(\sqrt[3]{3\sqrt{3}+1} + \sqrt[3]{3\sqrt{3}-1}\right)^2}{3\sqrt[3]{2}}\cdot\sqrt[6]{13}$.

$$F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-11\pi}$$
when $x + \frac{1}{x} = \frac{\sqrt[3]{9\sqrt{3}+1} + \sqrt[3]{9\sqrt{3}-1}}{\sqrt{3}}\cdot\sqrt[6]{\frac{11}{2}}$

$$x^{\frac{3}{8}}\left\{\sqrt{\frac{\sqrt{x}+\frac{1}{\sqrt{x}}+1}{2}} - \sqrt{\frac{\sqrt{x}+\frac{1}{\sqrt{x}}-1}{2}}\right\}$$
$$\doteq \left\{\sqrt{A+1} \pm \sqrt{A}\right\} \text{ where } A = \frac{3}{2}\cdot\frac{1}{\sqrt{\frac{x+\frac{1}{x}+1}{x+\frac{1}{x}-2}}-1}$$

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$$\psi^2(x) + x\,\psi^2(x^5) = \frac{f(x^5)\,\phi(x^5)}{\chi(x)}.$$

$$\psi^2(x) + 5x\,\psi^2(x^5) = \frac{\phi^2(x)}{\chi(x)\,\chi(x^5)}$$

III. If $\alpha = \dfrac{1 \pm \sqrt{1-x^3}}{2}$ & $\beta = \dfrac{1-\sqrt{1-y^3}}{2}$

then $y\sqrt{2} = \sqrt[3]{x}\left\{\sqrt{1+x+x^2} - x \pm \sqrt{(1-x)(2\sqrt{1+x+x^2}-1\ldots} \pm \sqrt{(1-x)(2\sqrt{1+x+x^2}-1-2x)}\right\}$

$$F\left\{\frac{32+9\sqrt{7}-7\sqrt{3}-(16-\sqrt{21})\sqrt{2(\sqrt{21}-3)}}{64}\right\} = e^{-3\pi\sqrt{7}}$$

$$F\,\frac{1-\sqrt{1-(\sqrt{5}-2)^6(4\pm\sqrt{15})^4}}{2} = e^{-3\pi\sqrt{5}}$$

$$F\,\frac{1-\sqrt{1-\dfrac{4}{\left(\sqrt[3]{1+\sqrt{\frac{11}{27}}}+\sqrt[3]{1-\sqrt{\frac{11}{27}}}\right)^{12}}}}{2} = e^{-\pi\sqrt{11}}$$

$$F\,\frac{1-\sqrt{1-\dfrac{4}{\left(\sqrt[3]{1+\sqrt{\frac{19}{27}}}+\sqrt[3]{1-\sqrt{\frac{19}{27}}}\right)^{24}}}}{2} \quad e^{-\pi\sqrt{\ldots}}$$

$$F\,\frac{1-\sqrt{1-\dfrac{2^6}{\left(\sqrt[3]{1+\sqrt{\frac{19}{27}}}+\sqrt[3]{1-\sqrt{\frac{19}{27}}}\right)^{24}}}}{2}$$

$$F\left(\frac{1-\sqrt{1-\frac{\left(\sqrt[3]{1+\sqrt{\frac{31}{27}}}+\sqrt[3]{1-\sqrt{\frac{31}{27}}}\right)^{24}}{2^{4}}}}{2}\right)=e^{-\pi\sqrt{31}}$$

XVII $$\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}+\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

$$-2\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}\right\}=\frac{1+(\tfrac{1}{2})^{2}\alpha+\cdots}{1+(\tfrac{1}{2})^{2}\beta+\cdots}$$

$$F\left(\frac{1-\sqrt{1-\left(\frac{1}{\sqrt{5-\sqrt{17}}}-\frac{1}{\sqrt{3+\sqrt{17}}}\right)^{24}}}{2}\right)=e^{-\pi\sqrt{17}}$$

XXXIII ? $$\sqrt{\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}+\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[8]{\frac{\beta}{\alpha}}+\cdots\right\}}$$

$$-2\sqrt[16]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\sqrt[4]{\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}+\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}}$$

$$=\sqrt{\frac{1+(\tfrac{1}{2})^{2}\alpha+(\tfrac{1\cdot 3}{2\cdot 4})^{2}\alpha^{2}+\cdots}{1+(\tfrac{1}{2})^{2}\beta+(\tfrac{1\cdot 3}{2\cdot 4})^{2}\beta^{2}+\cdots}}$$

IX $$\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}+\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

$$-4\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}\right\}=\left\{\frac{1+(\tfrac{1}{2})^{2}\alpha+\cdots}{1+(\tfrac{1}{2})^{2}\beta+\cdots}\right.$$

VII $$\left\{\sqrt[4]{\frac{(1-\beta)^{7}}{1-\alpha}}+\sqrt[4]{\frac{\beta^{7}}{\alpha}}+1\right\}-2\left\{\sqrt[8]{\frac{\beta^{7}(1-\beta)^{7}}{\alpha(1-\alpha)}}+\sqrt[8]{\frac{(1-\beta)^{7}}{1-\alpha}}+\sqrt[8]{\cdots}\right.$$

$$-\sqrt[24]{\frac{\beta^{7}(1-\beta)^{7}}{\alpha(1-\alpha)}}\left\{\sqrt[8]{\frac{(1-\beta)^{7}}{1-\alpha}}+\sqrt[8]{\frac{\beta^{7}}{\alpha}}+1\right\}-3\sqrt[12]{\frac{\beta^{7}(1-\beta)^{7}}{\alpha(1-\alpha)}}=0$$

~~[illegible] of~~ $2x\,\psi\left(x^{\frac{8m}{[illegible]}}\right)\psi\left(x^{[illegible]}\right)$ [illegible] $=\phi([illegible])$

[illegible]

VI $\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} = \sqrt{1-\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}$

V $\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3} = \sqrt{1-\sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}}$ $\Big\} = \sqrt{\frac{1+\ldots}{2}}$ 149

III $$\frac{1+\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{\sqrt{1-\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}} = \frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}$$

VII $$\frac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\frac{\beta^7}{\alpha}}}{1-\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}} = \frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}$$

$$\frac{\phi^2(e^{-7\pi})}{\phi^2(e^{-\pi})} = \frac{\left(\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}\right)\sqrt[8]{28}}{~~\sqrt[8]{7}~~\ 14}$$

$$\phi(x^3)\phi(x^5) = \phi(-x^2)\phi(-x^{30}) + 2x^2\psi(x)\psi(x^{15}).$$

If I α, 3rd β, 5th γ & 15th δ then

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

$$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$$\sqrt[8]{\alpha\delta} - \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt[8]{\beta\gamma} - \sqrt[8]{(1-\beta)(1-\gamma)}$$

$$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt[3]{4}\cdot\sqrt[24]{\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}} - 1$$

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = -\sqrt[3]{4}\cdot\sqrt[24]{\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}} + 1.$$

VII $$\frac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - 1}{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\frac{\beta^7}{\alpha}}} = \sqrt[8]{\alpha\beta}.$$

1st α, 5th β, 7th γ & 35th δ

$$\sqrt[4]{\alpha\delta} + \sqrt[4]{(1-\alpha)(1-\delta)} + 2\sqrt[3]{2}\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)}$$

$$+ \sqrt[4]{\beta\gamma} + \sqrt[4]{(1-\beta)(1-\gamma)} + 2\sqrt[3]{2}\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}$$

$$= 1 + \left\{1 + 2\sqrt[3]{2}\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right\}^2.$$

$$P = \sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} + \sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

$$Q = \sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{\sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} + 1\right\}$$

$$R = \sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

17. $\sqrt{P^2 - 4Q} = K.$

47. $P + R^{\frac{1}{3}} - \sqrt{4Q + 4PR^{\frac{1}{3}} + 13R^{\frac{2}{3}}} = K.$

V
$$\frac{m + 2(m-n)\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}} + n\sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}}{m - n\sqrt[8]{16\alpha\beta(1-\alpha)(1-\beta)}}$$

$$= \frac{1 - \sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}} - \sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}}{\sqrt{1 - 3\sqrt[8]{16\alpha\beta(1-\alpha)(1-\beta)} + \sqrt[3]{16\alpha\beta(1-\alpha)(1-\beta)}}} = \frac{1 + (\frac{1}{2})^2\alpha + \cdots}{1 + (\frac{1}{2})^2\beta + \cdots}$$

XI
$$\frac{1 + 8\sqrt[3]{2}\sqrt[24]{\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}}}{\left\{\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\}\left\{\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)} + \sqrt{2 + 2\sqrt{\alpha\beta} + 2\sqrt{(1-\alpha)(1-\beta)}}\right.}$$

$$= \frac{1 + (\frac{1}{2})^2\alpha + (\frac{1\cdot 3}{2\cdot 4})^2\alpha^2 + \cdots}{1 + (\frac{1}{2})^2\beta + \cdots}$$ [illegible]

VII
$$\sqrt[8]{\frac{(1-\alpha)^7}{1-\beta}} + \sqrt[8]{\frac{\alpha^7}{\beta}} + 2\sqrt[24]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}$$

[illegible] $\left\{1 + \ldots\right\}^2$ [illegible]

= [illegible]

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1st α, 5th β, 7th γ & 35th δ

$$\left\{\sqrt[4]{\alpha\delta}+\sqrt[4]{(1-\alpha)(1-\delta)}+2\sqrt[3]{2}\,\sqrt[8]{\alpha\delta(1-\alpha)(1-\delta)}\right\}$$

$$\times\left\{\sqrt[4]{\beta\gamma}+\sqrt[4]{(1-\beta)(1-\gamma)}+2\sqrt[3]{2}\,\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}\right\}$$

$$=1-4\sqrt[3]{2}\,\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\left\{\sqrt[8]{\alpha\beta\gamma\delta}+\sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right\}$$

$$\phi(x)\phi(x^{27})-\phi(-x)\phi(-x^{27})=4xf(-x^6)f(-x^{18})+4x^7\psi(x^2)\psi(x^{54})$$

$$\phi(x)\phi(x^{35})-\phi(-x)\phi(-x^{35})=4xf(-x^{10})f(-x^{14})+4x^9\psi(x^2)\psi(x^{70})$$

$$\phi(x^5)\phi(x^7)-\phi(-x^5)\phi(-x^7)=4x^3\psi(x^{10})\psi(x^{14})-4x^3f(-x^2)f(-x^{70})$$

$$\phi(x)\phi(x^{63})-\phi(x^7)\phi(x^9)=2xf(x^3)f(x^{21})$$

If $\alpha,\beta,\gamma,\delta$ = 1, 3, 13, 39 or 1, 5, 11, 55 or 1, 7, 9, 63

Then
$$\frac{1+\sqrt[4]{(1-\alpha)(1-\delta)}+\sqrt[4]{\alpha\delta}}{1+\sqrt[4]{(1-\beta)(1-\gamma)}+\sqrt[4]{\beta\gamma}}=\frac{\sqrt[8]{(1-\alpha)(1-\delta)}-\sqrt[8]{\alpha\delta}}{\sqrt[8]{(1-\beta)(1-\gamma)}-\sqrt[8]{\beta\gamma}}$$

$$=\sqrt{\frac{1+(\tfrac14)^2\beta+\&c}{1+(\tfrac14)^2\alpha+\&c}\cdot\frac{1+(\tfrac14)^2\gamma+\&c}{1+(\tfrac14)^2\delta+\&c}}$$

$$=\frac{\sqrt[8]{\alpha\delta}\pm\sqrt[8]{\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{\beta\gamma}\mp\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}$$

($+$ in the 1st 2 cases; $-$ in 1, 7, 9, 63)

$$\psi(x^7)\psi(x^9)-x^6\psi(x)\psi(x^{63})=f(-x^6)f(-x^{42})$$

$$\phi(x)\phi(x^{135})=\phi(-x^{10})\phi(-x^{54})+2x^4\psi(x^5)\psi(x^{27})+2xf(x^9)f(\ldots)$$

$$\phi(x^5)\phi(x^{27})=\phi(-x^2)\phi(-x^{270})+2x^{17}\psi(x)\psi(x^{135})+2x^2f(x^3)f(\ldots)$$

III $m^2\sqrt{\alpha(1-\alpha)} + \sqrt{\beta(1-\beta)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$

V $m\sqrt[4]{\alpha(1-\alpha)} + \sqrt[4]{\beta(1-\beta)} = \sqrt[3]{4}\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$.

III $\begin{cases} m\sqrt{1-\alpha} + \sqrt{1-\beta} = 2\sqrt[8]{(1-\alpha)(1-\beta)} \\ m\sqrt{\alpha} - \sqrt{\beta} = 2\sqrt[8]{\alpha\beta} \end{cases}$

II $m\sqrt{1-\alpha} + \sqrt{\beta} = 1$ }
IV $\sqrt{m}\,\sqrt[4]{1-\alpha} + \sqrt[4]{\beta} = 1$ }

II $m^2\sqrt{1-\alpha} + \beta = 1$
IV $m\sqrt[4]{1-\alpha} + \sqrt{\beta} = 1$
VIII $\sqrt{m}\,\sqrt[8]{1-\alpha} + \sqrt[4]{\beta} = 1$

II $m^2 = 2\cdot\frac{1+\beta}{1+(1-\alpha)} = 2\cdot\frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}$

IV $m = 2\cdot\frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}} = 2\cdot\frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}$.

XVI $\sqrt{m} = 2\cdot\frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}$.

IV $\sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{4}{m}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = m$.

VII $\sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{7}{m}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 3\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$
$= m$.

IX $m = 3\cdot\frac{1+2\sqrt[3]{\beta}}{1+2\sqrt[3]{1-\alpha}}$.

Equations for $1+\frac{1.3}{4^2}x+\frac{1.3.5.7}{4^2.8^2}x^2+\&c$ 151

(3) $\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}-\frac{9}{m^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=m^2$

(5) $\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-\frac{5}{m}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=m$

(9) $\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}-\frac{3}{\sqrt{m}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}=\sqrt{m}.$

(7) $\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}-\frac{49}{m^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-8\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[6]{\frac{\beta}{\alpha}}+\sqrt[6]{\frac{1-\beta}{1-\alpha}}\right)=m^2$

(13) $\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}-\frac{13}{m}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-4\sqrt[12]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[12]{\frac{\beta}{\alpha}}+\sqrt[12]{\frac{1-\beta}{1-\alpha}}\right)=m$

(25) $\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}-\frac{5}{\sqrt{m}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[24]{\frac{\beta}{\alpha}}+\sqrt[24]{\frac{1-\beta}{1-\alpha}}\right)=\sqrt{m}$

(2). $1+\frac{1.3}{4^2}\left\{1-\left(\frac{1-t}{1+3t}\right)^2\right\}+\&c$

$=\sqrt{1+3t}\left\{1+\frac{1.3}{4^2}t^2+\frac{1.3.5.7}{4^2.8^2}t^4+\right\}$

(3) If $\alpha=\frac{64t}{(9+6t-t^2)^2}$ $\beta=\frac{64t^3}{(27-18t-t^2)^2}$

then $\sqrt{1-\frac{2}{3}t-\frac{t^2}{27}}\left\{1+\frac{1.3}{4^2}\alpha+\right\}$.

$=\sqrt{1+2t-\frac{t^2}{3}}\left\{1+\frac{1.3}{4^2}\beta+\right\}$.

$1+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)=\phi^2(x)\phi^2(x^3).\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}$

$1+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)=\phi^2(x)\phi^2(x^7).\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}$

$$P = 1 - \sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}$$

$$Q = 4\left\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

$$R = 4\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}.$$

7. $P = 0$. 23. $P - R^{\frac{1}{3}} = 0$.

39. ~~$P^5 - R(6P^2+Q) + \frac{R^2}{P} + \frac{R^2P^2}{P^3-R} = 0$.~~

55. ~~$P^7 - R(18P^4+9P^2Q+Q^2)$~~ $+ 10R^2P + \frac{QR^2}{P} + \frac{R^2P^2Q}{P^3-R} = 0$.

71. $P^3 - R^{\frac{1}{3}}(4P^2+Q) + 2PR^{\frac{2}{3}} - R = 0$.

~~87. $P^{11} -$~~

15. $P^2 - Q + \frac{R}{P} = 0$.

31. $P^2 - Q - \sqrt{PR} = 0$

47. $P^2 - Q - PR^{\frac{1}{3}} - 2R^{\frac{2}{3}} = 0$.

~~63~~. 95. $(P^2-Q)^2 - PR^{\frac{1}{3}}(5P^2-4Q) + QR^{\frac{2}{3}} - R\left(P+\frac{Q}{P}\right) - 5R^{\frac{4}{3}} = 0$

3. $P = 0$ 11. $P - R^{\frac{1}{3}} = 0$. 19. $P^5 - 7P^2R - QR = 0$

35. $P^3 - R^{\frac{1}{3}}(5P^2+Q) + 2R^{\frac{2}{3}}P - R - \frac{R^{\frac{4}{3}}}{P} = 0$.

~~51. P^5~~ 27. $P^7 - R(29P^4+11P^2Q+Q^2) - 17R^2P - 3\frac{R^2}{P^2}(PQ+R) = 0$

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7, 23 &c
$$\begin{cases} P = x f(x^2, x^{14}) - x^{\frac{n+1}{8}} \psi(x^4) \\ Q = \{\phi(-x^2) + 4 x^{\frac{n+9}{8}} f(x^2, x^{14})\} \psi(x^4). \\ R = x^{\frac{n+1}{8}} f^3(-x^4). \end{cases}$$

3, 11 &c
$$\begin{cases} P = x \psi(x^8) - x^{\frac{n+1}{4}} \psi(x^4) \\ Q = \phi^2(-x^4) + 16 x^{\frac{n+5}{4}} \psi(x^2) \psi(x^8). \\ R = x^{\frac{n+1}{4}} f^3(-x^2). \end{cases}$$

1, 5 &c
$$\begin{cases} P = \{x \psi(x^4) - 2 x^{\frac{n+1}{2}} \phi(x^2)\} \psi(x^4). \\ Q = \phi^4(-x^2) + 128 x^{\frac{n+3}{2}} \psi^2(x^2) \psi^2(x^4). \\ R = x^{\frac{n+1}{2}} f^6(-x^4). \end{cases}$$

15, 31 &c.
$$\begin{cases} P = f(x^6, x^{10}) + x^{\frac{n+1}{8}} \psi(x^4). \\ Q = \{\phi(-x^2) + 4 x^{\frac{n+1}{8}} f(x^6, x^{10})\} \psi(x^4) \\ R = x^{\frac{n+1}{8}} f^3(-x^4). \\ P_2 = P^2 - Q = x^2 f^2(x^2, x^{14}) - 2 x^{\frac{n+1}{8}} f(x^6, x^{10}) \\ \quad \times \psi(x^4) + x^{\frac{n+1}{4}} \psi^2(x^4). \end{cases}$$

$P = 1 + \sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)}$ | $1 - \sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}$

$Q = 4\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} + \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\}$ | 16.

$R = 4 \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$ | 16.

$P = 1 - \sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}$ | $1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}$

$Q = 4.$ | 64.

$R = 1.$ | 32.

. $P - R^{\frac{1}{3}} = 0.$ 9. $P^5 - R(14P^2 + Q) - \frac{3R^2}{P} = 0.$

13. $\sqrt{P}\,(P^3 + 8R) - \sqrt{R}\,(11P^2 + Q) = 0.$

17. $P^3 - R^{\frac{1}{3}}(10P^2 + Q) + 28R^{\frac{2}{3}}P + 12R = 0.$

29. $\sqrt{P}\,(P^2 + 17R^{\frac{1}{3}}P - 9R^{\frac{2}{3}})$
$- \sqrt[6]{R}\,(9P^2 + Q - 13R^{\frac{1}{3}}P + 15R^{\frac{2}{3}}) = 0.$

9. $P^5 - R^{\frac{1}{3}}(14P^4 + 9P^2Q + Q^2) + R^{\frac{2}{3}}P(19P^2 + Q)$
$+ 6R(7P^2 + Q) + 4R^{\frac{4}{3}}P - 3R^{\frac{5}{3}} = 0.$

21.

153

$$= \phi^2(x)\,\phi^2(x\ldots) \qquad \frac{3+\sqrt{\ldots}+\sqrt{(1-\ldots)(1-\ldots)}}{4} \qquad \int \frac{\ldots}{2}$$

$$F\ \frac{1-\sqrt{1-\frac{1}{4}(\sqrt[3]{2}-1)^8}}{2} = e^{-3\pi\sqrt{3}}$$

$$F\ \frac{1-\sqrt{1-(\sqrt{37}-6)^6}}{2} = e^{-\pi\sqrt{37}}$$

$$F\ \frac{1-\sqrt{1-4\cdot\left(\frac{\sqrt{13}-3}{8}\right)^4\left(\sqrt{\frac{5+\sqrt{13}}{8}}\pm\sqrt{\frac{\sqrt{13}-3}{8}}\right)^2}}{2} = e^{-\pi\sqrt{39}}$$

$$F\ \frac{1-\sqrt{1-\left(\sqrt{\frac{13+\sqrt{97}}{8}}-\sqrt{\frac{5+\sqrt{97}}{8}}\right)^{24}}}{2} = e^{-\pi\sqrt{97}}$$

~~$$F\ \frac{1-\sqrt{1-\left(\sqrt{\frac{15+\sqrt{193}}{4}}-\sqrt{\frac{11+\sqrt{193}}{4}}\right)^{24}}}{2} = e^{-\pi\sqrt{\ldots}}$$~~

$$F\ \frac{1-\sqrt{1-\frac{1}{4}\cdot\left(\frac{5-\sqrt{21}}{4}\right)^4\left\{\sqrt{\frac{5+\sqrt{21}}{8}}\pm\sqrt{\frac{\sqrt{21}-3}{8}}\right\}^{24}}}{2} = e^{-\pi\sqrt{63}}$$

$\phi(-x)\phi(-x^{95}) + 4x f(x^5) f(x^{19})$

13 $= \phi(x)\phi(x^{95}) + 4x^6 f(-x^3, -x^5) f(-x^{285}, -x^{475})$

17 $\phi(-x)\phi(-x^{63}) + 4x f(x^3) f(x^{21})$

2 $= \phi(x)\phi(x^{63}) + 4x^4 f(-x^3, -x^5) f(-x^{189}, -x^{315})$.

$\phi(-x)\phi(-x^{143}) + 4x f(x^{11}) f(x^{13})$

$= \phi(x)\phi(x^{143}) - 4x^9 f(-x^3, -x^5) f(-x^{429}, -x^{715})$.

7. $\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}} + \sqrt[4]{\frac{\beta^3}{\alpha}} - \sqrt[4]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} = \frac{m^2}{2}\left\{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right\}$

$\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} - \sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} \cancel{= \frac{m}{2}\left\{1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right\}}$

$= m\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$.

$m - \frac{3}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)$

$m - \frac{5}{m} = 4\left(\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}\right) \Big/ \sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$

$m - \frac{7}{m} = 2\left(\sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}\right)\left(2+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right)$

$m - \frac{11}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)\left(4+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right)$

7: — $\cos 2u = (2\cos u - 1)\sqrt{4\cos u - 3}$

$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} = 1 - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$

1, 3, 5, 15

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} = -\sqrt{\frac{1+(\ )^2\alpha}{1+(\ )^2\beta}\cdot\frac{1+(\ )}{1+(\ )}}$$

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = \sqrt{\frac{1+(\ )^2\beta}{1+(\ )^2\alpha}\cdot\frac{1+(\ )^2\gamma}{1+(\ )^2\delta}}$$

$$\sqrt[8]{\alpha\beta\gamma\delta} + \sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} + \sqrt[3]{2}\cdot\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} = 1$$

1, 5, 25

$$\frac{1+\sqrt[3]{4}\cdot\sqrt[24]{\frac{\beta^{10}(1-\beta)^{10}}{\alpha\gamma(1-\alpha)(1-\gamma)}}}{1+\sqrt[3]{4}\cdot\sqrt[24]{\frac{\alpha^5\gamma^5(1-\alpha)^5(1-\gamma)^5}{\beta^2(1-\beta)^2}}} = \frac{\phi^2(x)\,\phi^2(x^{25})}{\phi^4(x^5)},$$

1, 5, 7, 35.

1, 3, 13, 39 or 1, 5, 7, 35

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + 2\sqrt[12]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

+ in 13 α − in 7.

$$= \pm\sqrt{\frac{1+(\ )^2\alpha+\ }{1+(\ )^2\beta+\ }\cdot\frac{1+(\ )^2\delta+\ }{1+(\ )^2\gamma+\ }}$$

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 2\sqrt[12]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+(\ )^2\beta+\ }{1+(\ )^2\alpha+\ }\cdot\frac{1+(\ )^2\gamma+\ }{1+(\ )^2\delta+\ }}$$

1, 3, 9

$$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}} - \sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} = -3\cdot\frac{1+(\ )^2\alpha}{1+(\ )^2\beta}\cdot\frac{1+(\ )^2\gamma+}{1+(\ )^2}$$

$$\sqrt[4]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} - \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} = \frac{1+(\ )^2\beta+}{1+(\ )^2\alpha+}\cdot\frac{1+(\ )^2\beta+}{\ }$$

1, 3, 7, 21.

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + 4\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$$\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 4\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

1, 3, 9, 27

$$\left[\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} + \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 2\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}\left\{1+\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}}+\sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}\right\}\right] = -3\cdot\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\times\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}$$

$$\left[\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} + \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} - 2\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}\left\{1+\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}}+\sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}}\right\}\right] = \frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\times\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}$$

1, 5, 25

$$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}} + \sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} - 2\sqrt[8]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}}\times\left\{1+\sqrt[8]{\frac{\beta^2}{\alpha\gamma}}+\sqrt[8]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}}\right\} = 5\cdot\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\beta+\&c}$$

$$\sqrt[4]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} + \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} - 2\sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}\times\left\{1+\sqrt[8]{\frac{\alpha\gamma}{\beta^2}}+\sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}}\right\} =$$

1, 5, 25

$$\sqrt[8]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} + \sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}$$

$$= \frac{\{1+(\tfrac{1}{2})^2\beta + \ldots\}^2}{\sqrt{\{1+(\tfrac{1}{2})^2\alpha+\ldots\}\{1+(\tfrac{1}{2})^2\gamma+\ldots\}}}$$

1, 3, 9, 27

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} + \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+(\tfrac{1}{2})^2\beta+}{1+(\tfrac{1}{2})^2\alpha+} \cdot \frac{1+(\tfrac{1}{2})^2\gamma+}{1+(\tfrac{1}{2})^2\delta+}}$$

N.B. For 1, 5, 7, 35 same as for 1, 3, 11, 33 $4R^{\frac{2}{3}}$ instead of $4R^{\frac{1}{3}}$

For 1, 3, 9, 27 :- $P^5 - R(11P^2+Q) + 9\frac{R^2}{P} + 6\frac{R^3}{P^4} = 0.$

1, 7, 21, 3 or 1, 3, 33, 11.

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 2\sqrt[12]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+(\tfrac{1}{2})^2\alpha}{1+(\tfrac{1}{2})^2\beta} \cdot \frac{1+(\tfrac{1}{2})^2\delta}{1+(\tfrac{1}{2})^2\gamma}}$$

1, 3, 15, 5

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 4\sqrt[6]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \frac{1+(\tfrac{1}{2})^2\alpha}{1+(\tfrac{1}{2})^2\beta} \cdot \frac{1+(\tfrac{1}{2})^2\delta}{1+(\tfrac{1}{2})^2\gamma}$$

For 1, 3, 7, 21 the same as 1, 3, 3, 9 ; $P^2 = Q + \frac{3R}{P}$

1, 5, 5, 25 ; $P^3 - R^{\frac{1}{3}}(5P^2+Q) - R^{\frac{2}{3}}P + 3R - \frac{R^{\frac{4}{3}}}{P} = 0$

1, 3, 17, 51 ; $P^3 - R^{\frac{1}{2}}(7P^2+Q) + 13R^{\frac{2}{3}}P - 12R = 0.$

1, 5, 11, 55 ; $P^3 - R^{\frac{1}{3}}(4P^2+Q) - R^{\frac{2}{3}}P + 4R = 0.$

$$37.\quad P^3 - R^{\frac{1}{3}}(7P^2 + Q) - 3PR^{\frac{2}{3}} - 25R - M(19P^2 - 8c) + 8c$$

1, 4, 5, 20 or 1, 2, 7, 14

$$\frac{1 + 2\{\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)}\}}{1 + 2\{\sqrt[3]{\beta\gamma} + \sqrt[3]{(1-\beta)(1-\gamma)}\}} = \frac{1 + \frac{1.2}{3^2}\beta + \&c}{1 + \frac{1.2}{3^2}\alpha + \&c} \cdot \frac{1 + \frac{1.2}{3^2}\gamma + \&c}{1 + \frac{1.2}{3^2}\delta + \&c}$$

1, 2, 4, 8

$$\frac{1 - \sqrt[3]{\alpha\delta} - \sqrt[3]{(1-\alpha)(1-\delta)}}{3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)}} = \frac{1 + \frac{1.2}{3^2}\beta + \&c}{1 + \frac{1.2}{3^2}\alpha + \&c} \cdot \frac{1 + \frac{1.2}{3^2}\gamma + \&c}{1 + \frac{1.2}{3^2}\delta + \&c}$$

$\sqrt{2}$ $\quad F(\sqrt{2}-1)^2 = e^{-\pi\sqrt{2}}$

16. $\quad (2-\sqrt{3})^2(\sqrt{3}\pm\sqrt{2})^2$

$\sqrt{10}$ $\quad (\sqrt{10}-3)^2(3\pm\sqrt{8})^2$

$\sqrt{18}$ $\quad (5\sqrt{2}-7)^2(7\pm 4\sqrt{3})^2$

$\sqrt{22}$ $\quad$ ~~$(\sqrt{66}-\sqrt{65})^2(\sqrt{65}-\sqrt{64})^2$~~ $(10-3\sqrt{11})^2(3\sqrt{11}\pm 7\sqrt{2})^2$

$\sqrt{70}$ $\quad (6-\sqrt{35})^2(15-4\sqrt{14})^2(8-3\sqrt{7})^2(3\sqrt{14}-5\sqrt{5})^2$

$\sqrt{58}$ $\quad (13\sqrt{58}-99)^2(99\pm 70\sqrt{2})^2$.

$\sqrt{30}$ $\quad (4-\sqrt{3})^2(5-2\sqrt{6})^2(4-\sqrt{15})^2(\sqrt{6}-\sqrt{5})^2$

Let $\alpha\beta = 16t(1-t)^3$ & $(1-\alpha)(1-\beta) = 16t^3(1-t)$

$\frac{p(2-p)}{1+2p} = 2t$. then $F\left(p^3 \cdot \frac{2+p}{1+2p}\right) = e^{-\pi\sqrt{3\cdot 2}}$

$4t(1-t) = k$

$$\frac{1-\sqrt{1-4k}}{2}$$

$(1-2k\pm\sqrt{1-k\cdot 1-4k})^2 = e^{-\pi\sqrt{5}}$

$$F\left(\frac{1-\sqrt{1-(10\pm 3\sqrt{11})^2(2-\sqrt{3})^6}}{2}\right) = e^{-\pi\sqrt{33}}.$$

$$F\left(\frac{1-\sqrt{1-\frac{\left(\frac{\sqrt{5}+1}{2}\sqrt[3]{10}+\frac{\sqrt{5}-1}{2}\sqrt[3]{4\sqrt{5}}-\sqrt{5}-1\right)^{24}}{3^{24}\cdot 2^{10}}}}{2}\right)$$

$$= e^{-5\pi\sqrt{3}}.$$

$$F\left(\frac{1+\sqrt{1-\frac{\left(\frac{\sqrt{5}+1}{2}\sqrt[3]{4\sqrt{5}}-\frac{\sqrt{5}-1}{2}\sqrt[3]{10}+\sqrt{5}-1\right)^{24}}{3^{24}\cdot 2^{10}}}}{2}\right)$$

$$= e^{-\frac{\pi}{5}\sqrt{3}}.$$

$$\sqrt[8]{1\pm\sqrt{1-(\sqrt{5}-2)^8}} = \frac{\sqrt{5}-1}{2}\cdot\frac{\sqrt[4]{5}\pm 1}{\sqrt{2}}$$

$$\sqrt[8]{1\pm\sqrt{1-(2-\sqrt{3})^4}} = \frac{\sqrt{2}\cdot\sqrt[4]{3}\pm(\sqrt{3}-1)}{2\sqrt[4]{2}}.$$

If $a=\sqrt[4]{60}$, $b = 2-\sqrt{3}+\sqrt{5}$ & $2c = 1+\frac{a+b}{a-b}\sqrt{5}$

then $(\sqrt{c^2+1}-c)\sqrt[5]{e^{6\pi}} = \frac{1}{1+}\frac{e^{-6\pi}}{1+}\frac{e^{-12\pi}}{1+}$ &c

If $2c = 1+\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\sqrt{5}$ then $\frac{e^{-\frac{4\pi}{5}}}{1+}\frac{e^{-4\pi}}{1+}\frac{e^{-8\pi}}{1+}$ &c

If $a = 3+\sqrt{2}-\sqrt{5}$ & $b = \sqrt[4]{20}$. then $\frac{e^{-\frac{8\pi}{5}}}{1+}\frac{e^{-8\pi}}{1+}\frac{e^{-16\pi}}{1+}$ &c

If $a = \sqrt[4]{5}(2-\sqrt{2})$ & $b = 1+\sqrt{2}+\sqrt{5}$
$-\sqrt[4]{2}(3-\sqrt{2}+\sqrt{5}-\sqrt{10})$

then $\frac{e^{-\frac{16\pi}{5}}}{1+}\frac{e^{-16\pi}}{1+}\frac{e^{-32\pi}}{1+}$ &c

$$\phi(e^{-45\pi}) = \frac{3+\sqrt{5} + (\sqrt{3}+\sqrt{5}+\sqrt[4]{60})\sqrt[3]{2+\sqrt{3}}}{3\sqrt{10}+10\sqrt{5}}\,\phi(e^{-\pi})$$

$$\frac{\beta(1-\beta)}{1+\beta} = u \quad \text{then} \quad \tfrac{1}{4}\left(u+\tfrac{1}{u}-2\right)$$

$1 - 1$

$3 - 3$

$5 - 9$

$7 - (\sqrt{2}+1)^2(1+2\sqrt{2})$

$9 - 49$

$11 - 99$

$15 - 3(5+4\sqrt{2})^2$

$23 - 9(\sqrt{2}+1)^4(3+4\sqrt{2})$

$29 - 99^2$

$35 - 63(8\sqrt{2}+5\sqrt{5})^2$

$71 - 9(\sqrt{2}+1)^{10}(2\sqrt{2}+1)^2(6\sqrt{2}+1).$

$$F\ \frac{1-\sqrt{1-\frac{1}{4}\left(\frac{\sqrt{6+3\sqrt{3}}-\sqrt{2+3\sqrt{3}}}{2}\right)^{18}(\sqrt{3}+1)^6\left(2\sqrt{3\sqrt{3}+2}-3\sqrt{3\sqrt{3}-2}\right)^2}}{2}$$

$$= e^{-\pi\sqrt{69}}.$$

$$F\ \frac{1-\sqrt{1-4\left(\frac{\sqrt{6+3\sqrt{3}}-\sqrt{2+3\sqrt{3}}}{2}\right)^{6}\left(\frac{\sqrt{3}+1}{2}\right)^6\left(2\sqrt{3\sqrt{3}+2}+3\sqrt{3\sqrt{3}-2}\right)^2}}{2}$$

$$= e^{-\pi\sqrt{\frac{23}{3}}}.$$

$\sqrt{42}\quad (8-3\sqrt{7})^2(7-4\sqrt{3})^2(3-2\sqrt{2})^2(\sqrt{7}-\sqrt{6})^2.$

$\sqrt{78}\quad (2-\sqrt{3})^6(\sqrt{13}-2\sqrt{3})^4(3\sqrt{3}-\sqrt{26})^2(5-2\sqrt{6})^2.$

F $\frac{1-\sqrt{1-\left(\sqrt{\frac{9+\sqrt{73}}{8}}-\sqrt{\frac{1+\sqrt{73}}{8}}\right)^{24}}}{2} = e^{-\pi\sqrt{73}}$

~~F $\frac{1-\sqrt{1-\left(\sqrt{\frac{15+\sqrt{193}}{4}}-\sqrt{\frac{11+\sqrt{193}}{4}}\right)^{24}}}{2} = e^{-\pi\sqrt{193}}$~~

F $\frac{1-\sqrt{1-\frac{\left(\sqrt[3]{1+\sqrt{\frac{43}{27}}}+\sqrt[3]{1-\sqrt{\frac{43}{27}}}\right)^{24}}{2^{10}}}}{2} = e^{-\pi\sqrt{43}}$

$$1+\frac{x^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}+\frac{2x^{\frac{2}{5}}}{1-x^{\frac{2}{5}}}+\&c$$
$$-25\left(\frac{x^5}{1-x^5}+\frac{2x^{10}}{1-x^{10}}+\&c\right)$$
$$=\frac{f^5(-x)}{f(-x^5)}\sqrt{1+2x^{\frac{1}{5}}\frac{f(-x^5)}{f(-x^{\frac{1}{5}})}+5x^{\frac{2}{5}}\frac{f^2(-x^5)}{f^2(-x^{\frac{1}{5}})}}$$

$\sqrt{102}$ $\left(\frac{\sqrt{51}-7}{\sqrt{2}}\right)^4(5-2\sqrt{6})^4(\sqrt{51}-5\sqrt{2})^2(2-\sqrt{3})^4$.

$(5\sqrt{130}-57)^2(3-2\sqrt{2})^4(\sqrt{26}-5)^4(\sqrt{10}-3)^4$.

$\sqrt{130}$ $(2\sqrt{5}-\sqrt{19})^4(\sqrt{19}-3\sqrt{2})^4$

$\sqrt{190}$ $\left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4(37\sqrt{19}-51\sqrt{10})^2$

$$F\left[\frac{1-\sqrt{1-\left(\sqrt{\frac{6+3\sqrt{3}}{4}}-\sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{12}t}}{2}\right] = e^{-\pi\sqrt{69}}$$

where $$t = \left(\sqrt{\frac{(5+3\sqrt{3})\sqrt{4+6\sqrt{3}}+1}{2}} \pm \sqrt{\frac{(5+3\sqrt{3})\sqrt{4+6\sqrt{3}}-1}{2}}\right)^{4}.$$

$$4\beta(1-\beta) = \left(\frac{\sqrt{13}-3}{2}\right)^{6}(\sqrt{13}-\sqrt{12})^{4}\left(\sqrt{\frac{11+6\sqrt{3}}{2}} \pm \sqrt{\frac{9+6\sqrt{3}}{2}}\right)^{8}$$ for $e^{-3\pi\sqrt{13}}$.

$$4\beta(1-\beta) = (\sqrt{37}-6)^{6}(2\sqrt{37}-7\sqrt{3})^{4}\left(\sqrt{73+42\sqrt{3}} \pm \sqrt{72+42\sqrt{3}}\right)^{8}$$ for $e^{-3\pi\sqrt{37}}$.

$$4\beta(1-\beta) = \left(\frac{\sqrt[3]{2(\sqrt{3}-1)}-1}{\sqrt[3]{2(\sqrt{3}+1)}+1}\right)^{8}$$ for $e^{-9\pi}$.

$$4\beta(1-\beta) = \frac{1}{4}\cdot\left\{\frac{1 \pm \left(2\sqrt[6]{\frac{27}{27}}-\sqrt{\frac{7}{3}}\right)}{2}\right\}^{24}$$ for $e^{-7\pi\sqrt{3}}$.

$$4\beta(1-\beta) = \frac{x^{24}}{2^{10}}$$ for $e^{-11\pi\sqrt{3}}$ where

$$x^3 + x^2(1 \pm \sqrt{11+2\sqrt{33}}) + x(4+\sqrt{33} \pm \sqrt{11+2\sqrt{33}}) - 2 = 0$$

$\sqrt{217}$ $$\left(\sqrt{\frac{11+4\sqrt{7}}{2}}-\sqrt{\frac{9+4\sqrt{7}}{2}}\right)^{12}\left(\sqrt{\frac{16+5\sqrt{7}}{4}} \pm \sqrt{\frac{12+5\sqrt{7}}{4}}\right)^{12}$$

$\sqrt{205}$ $$(\sqrt{5}-2)^{8}\left(\frac{3\sqrt{5}-\sqrt{41}}{2}\right)^{6}\left(\sqrt{\frac{7+\sqrt{41}}{8}} \pm \sqrt{\frac{\sqrt{41}-1}{8}}\right)^{24}$$

$\sqrt{265}$ $$\left(\frac{\sqrt{53}\pm 7}{2}\right)^{6}(\sqrt{5}\pm 2)^{6}\left(\sqrt{\frac{89+5\sqrt{265}}{8}}-\sqrt{\frac{81+5\sqrt{265}}{8}}\right)^{12}$$

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$\sqrt{57}.\quad \frac{1}{4}(3\sqrt{19}-13)^4(2\pm\sqrt{3})^6$

$\sqrt{93}.\quad \frac{1}{4}(39-7\sqrt{31})^4\left(\frac{\sqrt{31}\pm3\sqrt{3}}{2}\right)^6.$

$\sqrt{177}\quad \frac{1}{4}(3\sqrt{59}\pm23)^4(2-\sqrt{3})^{18}.$

$\sqrt{85}\quad (\sqrt{5}\pm2)^8\left(\frac{\sqrt{85}-9}{2}\right)^6.$

$\sqrt{133}.\quad (8-3\sqrt{7})^6\left(\frac{5\sqrt{7}\pm3\sqrt{19}}{2}\right)^6.$

$\sqrt{55}\quad \frac{1}{64}(\sqrt{5}-2)^4\left(\sqrt{\frac{7+\sqrt{5}}{8}}\pm\sqrt{\frac{\sqrt{5}-1}{8}}\right)^{24}$

$\sqrt{65}\quad \left(\frac{\sqrt{13}\pm3}{2}\right)^6.(\sqrt{5}\pm2)^2.\left(\sqrt{\frac{9+\sqrt{65}}{8}}-\sqrt{\frac{1+\sqrt{65}}{8}}\right)^{12}.$

$\sqrt{253}\quad (24-5\sqrt{23})^6\left(\frac{9\sqrt{23}\pm13\sqrt{11}}{2}\right)^6.$

$\sqrt{145}\quad (\sqrt{5}-2)^6\left(\frac{\sqrt{29}-5}{2}\right)^6\left(\sqrt{\frac{17+\sqrt{145}}{8}}\pm\sqrt{\frac{9+\sqrt{145}}{8}}\right)^{12}$

$\sqrt{117}.\quad \left(\frac{\sqrt{13}-3}{2}\right)^6(\sqrt{13}-2\sqrt{3})^4\left(\frac{\sqrt{4+\sqrt{3}}\pm\sqrt[4]{3}}{2}\right)^{24}.$

$\sqrt{333}\quad (\sqrt{37}-6)^6(2\sqrt{37}-7\sqrt{3})^4\left(\frac{\sqrt{7+2\sqrt{3}}\pm\sqrt{3+2\sqrt{3}}}{2}\right)^{24}.8$

$\sqrt{153}\quad \left(\sqrt{\frac{5+\sqrt{17}}{8}}-\sqrt{\frac{\sqrt{17}-3}{8}}\right)^{48}\left(\sqrt{\frac{37+9\sqrt{17}}{4}}\pm\sqrt{\frac{33+9\sqrt{17}}{4}}\right)^{12}$

$\sqrt{77}\quad (8\pm3\sqrt{7})^3\left(\frac{\sqrt{11}\pm\sqrt{7}}{2}\right)^3\left(\sqrt{\frac{6+\sqrt{11}}{4}}-\sqrt{\frac{2+\sqrt{11}}{4}}\right)^{12}$

$\sqrt{69}\quad \left(\frac{5\pm\sqrt{23}}{\sqrt{2}}\right)^2\left(\frac{3\sqrt{3}\pm\sqrt{23}}{2}\right)^3\left(\sqrt{\frac{6+3\sqrt{3}}{4}}-\sqrt{\frac{2+3\sqrt{3}}{4}}\right)$ [illegible]

$\sqrt{213}\quad \left(\frac{59\pm7\sqrt{71}}{\sqrt{2}}\right)^2\left(\frac{5\sqrt{3}\pm\sqrt{71}}{2}\right)^3\left(\sqrt{\frac{21+12\sqrt{3}}{2}}-\sqrt{\frac{19+12\sqrt{3}}{2}}\right)$ [illegible]

$(1-e^{-\pi\sqrt{n}})(1-e^{-3\pi\sqrt{n}})(1-e^{-5\pi\sqrt{n}})$ &c

$$= \sqrt[4]{2}\,.\, e^{-\frac{\pi}{24}\sqrt{n}}\,.\, g_n.$$

$g_2 = 1$; $g_6 = \sqrt[6]{1+\sqrt{2}}$; $g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}}$; $g_{14} = \sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}$

$g_{18} = \sqrt[3]{\sqrt{2}+\sqrt{3}}$; $g_{22} = \sqrt{1+\sqrt{2}}$; $g_{30} = \sqrt{\frac{1+\sqrt{5}}{2}}\,\sqrt[6]{3+\sqrt{10}}$.

$g_{58} = \sqrt{\frac{5+\sqrt{29}}{2}}$; $g_{70} = \frac{1+\sqrt{5}}{2}\sqrt{1+\sqrt{2}}$. $g_{46} = \sqrt{\frac{5+\sqrt{2}}{4}} + \sqrt{\frac{1+\sqrt{2}}{4}}$

$g_{142} = \sqrt{\frac{11+5\sqrt{2}}{4}} + \sqrt{\frac{7+5\sqrt{2}}{4}}$. $g_{82} = \sqrt{\frac{13+\sqrt{41}}{8}} + \sqrt{\frac{5+\sqrt{41}}{8}}$ 21

$$\sqrt{17.5}\,.\,\frac{\left\{\frac{\sqrt{5}-1}{2} + \sqrt[3]{\frac{5-\sqrt{5}}{4}}\left(\sqrt[3]{8-3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8-3\sqrt{5}-3\sqrt{21}}\right)\right\}^{24}}{64\,.\,3^{24}}$$

$$\sqrt{\frac{25}{7}}\,\frac{\left\{-\frac{\sqrt{5}+1}{2} + \sqrt[3]{\frac{5+\sqrt{5}}{4}}\left(\sqrt[3]{8+3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8+3\sqrt{5}-3\sqrt{21}}\right)\right\}}{64\,.\,3^{24}}$$

$\sqrt[32]{289}$. $\sqrt{\frac{17+\sqrt{17}+\sqrt{17\cdot 6}+42\sqrt{17}}{16}}$

$\sqrt{1+4\sqrt{17}+\sqrt{17\cdot 3+48}}$;

$g_{42} = \sqrt{\frac{7+\sqrt{3}}{2}}\,\sqrt[6]{3\sqrt{2}+\sqrt{7}}$. $g_{78} = \sqrt{\frac{3+\sqrt{13}}{2}}\,\sqrt[6]{5+\sqrt{26}}$

$g_{102} = \sqrt{1+\sqrt{2}}\,\sqrt[3]{3\sqrt{2}+\sqrt{17}}$. $g_{130} = \sqrt{\frac{3+\sqrt{13}}{2}}\,\sqrt{2+\sqrt{5}}$.

$g_{190} = \sqrt{2+\sqrt{5}}\,\sqrt{3+\sqrt{10}}$. $g_{34} = \sqrt{\frac{7+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-1}{8}}$

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$$\sqrt{289} \quad \left\{ \sqrt{\frac{17+\sqrt{17} + (5+\sqrt{17})\sqrt[?]{17}}{16}} + \sqrt{\frac{1+\sqrt{17} + (5+\sqrt{17})\sqrt[?]{17}}{16}} \right\}^{48}.$$

$$\sqrt{121}. \quad \left\{ \frac{\sqrt[3]{11-3\sqrt{11}}\left(\sqrt[3]{3\sqrt{11}+3\sqrt{3}-4} + \sqrt[3]{3\sqrt{11}-3\sqrt{3}-4}\right) - 2}{3\sqrt{2}} \right\}^{24}.$$

$$\sqrt{169} \quad \left\{ \frac{\sqrt[3]{\frac{13-3\sqrt{13}}{2}}\left(\sqrt[3]{3\sqrt{3}-\frac{11-\sqrt{13}}{2}} - \sqrt[3]{3\sqrt{3}+\frac{11-\sqrt{13}}{2}}\right) + (\sqrt{13}-2)}{3} \right\}^{24}$$

$$\sqrt{105} \quad \left(\frac{5-\sqrt{21}}{2}\right)^{6} (2 \pm \sqrt{3})^{6} (\sqrt{5} \pm 2)^{4} (6 \pm \sqrt{35})^{2}.$$

$$\sqrt{165} \quad (4-\sqrt{15})^{6} (3\sqrt{5} \pm 2\sqrt{11})^{4} \left(\frac{\sqrt{15} \pm \sqrt{11}}{2}\right)^{6} (\sqrt{5} \pm 2)^{4}.$$

$$\sqrt{345}. \quad \left(\frac{3\sqrt{3}-\sqrt{23}}{2}\right)^{12} \left(\frac{7\sqrt{23} \pm 15\sqrt{5}}{\sqrt{2}}\right)^{4} (\sqrt{5} \pm 2)^{8} (2 \pm \sqrt{3})^{6}$$

$$\sqrt{385} \quad (10-3\sqrt{11})^{6} (6 \pm \sqrt{35})^{6} \left(\frac{\sqrt{11} \pm \sqrt{7}}{2}\right)^{12} (\sqrt{5} \pm 2)^{8}.$$

$$\sqrt{273} \quad \left(\frac{15\sqrt{7}-11\sqrt{13}}{\sqrt{8}}\right)^{4} \left(\frac{\sqrt{13} \pm 3}{2}\right)^{12} \left(\frac{\sqrt{7} \pm \sqrt{3}}{2}\right)^{12} (2 \pm \sqrt{3})^{6}.$$

$$\sqrt{357} \quad \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{24} (8 \pm 3\sqrt{7})^{6} \left(\frac{11 \pm \sqrt{119}}{\sqrt{2}}\right)^{4} \left(\frac{\sqrt{21} \pm \sqrt{17}}{2}\right)^{6}.$$

$$g_{98} = \left(\sqrt{\frac{\sqrt{2} + 4 + \sqrt{14 + 4\sqrt{14}}}{8}} + \sqrt{\frac{\sqrt{2} - 4 + \sqrt{14 + 4\sqrt{14}}}{8}} \right)^2.$$

$$g_{90} = \sqrt{\frac{\sqrt{5}+1}{2}} \sqrt[6]{\sqrt{6}+\sqrt{5}} \left(\sqrt{\frac{3+\sqrt{6}}{4}} + \sqrt{\frac{\sqrt{6}-1}{4}} \right).$$

$$g_{198} = \sqrt{\sqrt{2}+1} \sqrt[6]{4\sqrt{2}+\sqrt{33}} \left(\sqrt{\frac{9+\sqrt{33}}{8}} + \sqrt{\frac{1+\sqrt{33}}{8}} \right)$$

If $g_n^4 - \frac{1}{g_n^4} = p$, then $g_{9n} = g_n \sqrt[6]{p + \sqrt{p^2+1}} \times$

$$\sqrt[3]{\left\{ \sqrt{\frac{p^2+4+\sqrt{(p^2+1)(p^2+4)}}{2}} + \sqrt{\frac{p^2+3+\sqrt{(p^2+1)(p^2+4)}}{2}} \right\}}.$$

$$g_{522} = \sqrt{\frac{5+\sqrt{29}}{2}} \sqrt[6]{5\sqrt{29}+11\sqrt{6}} \left(\sqrt{\frac{9+3\sqrt{6}}{4}} + \sqrt{\frac{5+3\sqrt{6}}{4}} \right)$$

$$g_{630} = \frac{1+\sqrt{5}}{2} \sqrt{1+\sqrt{2}} \sqrt[6]{\sqrt{15}+\sqrt{14}} \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}} \left(\sqrt{\frac{2+\sqrt{7}+\sqrt{15}}{4}} + \sqrt{\frac{\sqrt{7}+\sqrt{15}-2}{4}} \right)$$
$$\times \left(\sqrt{\frac{4+\sqrt{7}+\sqrt{15}}{8}} + \sqrt{\frac{\sqrt{7}+\sqrt{15}-4}{8}} \right).$$

$$g_{1170} = \sqrt{2+\sqrt{5}} \sqrt{\frac{3+\sqrt{13}}{2}} \sqrt{\sqrt{2}+\sqrt{3}} \sqrt[6]{2\sqrt{10}+\sqrt{39}}$$

$$g_{50} = \frac{1 + \sqrt[3]{\frac{5+\sqrt{5}}{4}} \left(\sqrt[3]{1+7\sqrt{5}+6\sqrt{6}} + \sqrt[3]{1+7\sqrt{5}-6\sqrt{6}} \right)}{3}.$$

$$g_{126} = \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}} \sqrt[6]{\sqrt{7}+\sqrt{6}} \left(\sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}} \right)^2.$$

$$g_{26} = \frac{1}{3} \left\{ \sqrt{2+\sqrt{13}} + \sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}} + \ldots} + [illegible] \right\}$$

[illegible]

$$g_{66} = \sqrt[4]{\sqrt2+\sqrt3}\,\sqrt[12]{3\sqrt{11}+7\sqrt2}\,\sqrt{\sqrt{\frac{7+\sqrt{33}}{8}} \pm \sqrt{\frac{\sqrt{33}-1}{8}}}.$$

$$g_{138} = \sqrt[4]{\frac{3\sqrt3+\sqrt{23}}{2}}\,\sqrt[12]{23\sqrt{23}+78\sqrt2}\,.\,\sqrt{\sqrt{\frac{5+2\sqrt6}{4}} \pm \sqrt{\frac{1+2\sqrt6}{4}}}.$$

$$g_{154} = \sqrt[4]{2\sqrt2\pm\sqrt7}\,\sqrt[4]{\frac{\sqrt{11}\pm\sqrt7}{2}}\,.\,\sqrt{\sqrt{\frac{9+2\sqrt{22}}{4}} \pm \sqrt{\frac{5+2\sqrt{22}}{4}}}.$$

$$g_{114} = \sqrt[4]{\sqrt3\pm\sqrt2}\,\sqrt[12]{\sqrt{19}\pm3\sqrt2}\,\sqrt{\frac{3\sqrt3+\sqrt{19}}{4}} + \sqrt{\frac{15+3\sqrt{57}}{8}}.$$

$$g_{238} = \left(\sqrt{\frac{5+3\sqrt2}{4}} + \sqrt{\frac{1+3\sqrt2}{4}}\right)\left(\sqrt{\frac{5+2\sqrt2}{4}} \pm \sqrt{\frac{1+2\sqrt2}{4}}\right).$$

$$g_{62} = \left(\sqrt{\frac{4+\sqrt{1+\sqrt2}+\sqrt{9+5\sqrt2}}{8}} + \sqrt{\frac{\sqrt{1+\sqrt2}+\sqrt{9+5\sqrt2}-4}{8}}\right)^2.$$

$$g_{94} = \left(\sqrt{\frac{4+\sqrt{7+\sqrt2}+\sqrt{7+5\sqrt2}}{8}} + \sqrt{\frac{\sqrt{7+\sqrt2}+\sqrt{7+5\sqrt2}-4}{8}}\right)^2$$

$$g_{154} = \sqrt[4]{2\sqrt2\pm\sqrt7}\,\sqrt[4]{\frac{\sqrt{11}\pm\sqrt7}{2}}\,\sqrt{\sqrt{\frac{13+2\sqrt{22}}{4}} + \sqrt{\frac{9+2\sqrt{22}}{4}}}$$

$$g_{310} = \frac{\sqrt5\pm1}{2}\,\sqrt{\sqrt2\pm1}\left(\sqrt{\frac{7+2\sqrt{10}}{4}} + \sqrt{\frac{3+2\sqrt{10}}{4}}\right)$$

$$g_{158} = \left(\sqrt{\frac{4+\sqrt{9+\sqrt2}+\sqrt{17+13\sqrt2}}{8}} + \sqrt{\frac{\sqrt{9+\sqrt2}+\sqrt{17+13\sqrt2}-4}{8}}\right)^2$$

$$\sqrt{465} \quad \left(\sqrt{\frac{13+2\sqrt{31}}{2}} - \sqrt{\frac{11+2\sqrt{31}}{2}}\right)^{12}\left(\frac{\sqrt{31}-3\sqrt3}{2}\right)^6(2-\sqrt3)^6$$

$$\times\left(\sqrt{\frac{6+\sqrt{31}}{4}} - \sqrt{\frac{2+\sqrt{31}}{4}}\right)^{12}(5\sqrt5-2\sqrt{31})^2(\sqrt5-2)^2$$

$$\sqrt{777} \quad (\sqrt{37}-6)^6(107\sqrt{37}-246\sqrt7)^2\left(\sqrt{\frac{17+6\sqrt7}{2}} - \sqrt{\frac{15+6\sqrt7}{2}}\right)^{12}$$

$$\times\left(\sqrt{\frac{10+3\sqrt7}{4}} - \sqrt{\frac{6+3\sqrt7}{4}}\right)^{12}(2-\sqrt3)^6\left(\frac{\sqrt7-\sqrt3}{2}\right)^6$$

$$\sqrt{1353} \quad \left(\sqrt{\frac{569+99\sqrt{33}}{8}} - \sqrt{\frac{561+99\sqrt{33}}{8}}\right)^{12}\left(\frac{321\sqrt{451}-6817}{\sqrt2}\right)^2$$

$$\times\left(\frac{\sqrt{123}-11}{\sqrt2}\right)^6(10-3\sqrt{11})^3(2-\sqrt3)^9\left(\sqrt{\frac{25+3\sqrt{33}}{8}} - \sqrt{17+ \ldots}\right)^{12}$$

$\sqrt{1645}$ $\left(\sqrt{\frac{751+41\sqrt{329}}{8}} - \sqrt{\frac{743+41\sqrt{329}}{8}}\right)^{12}\left(\frac{9\sqrt{329}}{2} - \frac{73\sqrt{5}}{}\right)^{3}$

$\times(\sqrt{5}-2)^{12}\left(\sqrt{\frac{127+7\sqrt{329}}{8}} - \sqrt{\frac{119+7\sqrt{329}}{8}}\right)^{12}\left(\frac{7-\sqrt{47}}{\sqrt{2}}\right)^{6}(8-3\sqrt{7})^{3}$

$\sqrt{897}$ $a + \frac{1}{a} = 58 + 9\sqrt{39}$ $s + \frac{1}{s} = 6 + \sqrt{39}$;

$\sqrt{1677}$ $s + \frac{1}{s} = 15 + 2\sqrt{43}$

$\sqrt{141}$ $(4\sqrt{3} \pm \sqrt{47})^{3}\left(\frac{7 \pm \sqrt{47}}{\sqrt{2}}\right)^{2}\left(\sqrt{\frac{18+9\sqrt{3}}{4}} - \sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{12}$

$\sqrt{445}$ $(\sqrt{5}-2)^{12}\left(\frac{\sqrt{445}-21}{2}\right)^{6}\left(\sqrt{\frac{13+\sqrt{89}}{8}} + \sqrt{\frac{5+\sqrt{89}}{8}}\right)^{24}$ 12

$\sqrt{553}$ $\left(\sqrt{\frac{143+16\sqrt{79}}{2}} - \sqrt{\frac{141+16\sqrt{79}}{2}}\right)^{12}\left(\sqrt{\frac{100+11\sqrt{79}}{4}} \pm \sqrt{\frac{96+11\sqrt{79}}{4}}\right)$

$g_{210} = \sqrt{\sqrt{3}+\sqrt{2}}\ \sqrt[6]{3\sqrt{14}+5\sqrt{5}}\ \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\ \sqrt{\frac{\sqrt{5}+1}{2}}$

$g_{330} = \sqrt{\sqrt{6}+\sqrt{5}}\ \sqrt{\frac{\sqrt{15}+\sqrt{11}}{2}} \cdot \frac{\sqrt{5}+1}{2} \cdot \sqrt[6]{\sqrt{11}+\sqrt{10}}$.

~~If u and v are~~ If $g_n^{6} = uv$; $u^2 + \frac{1}{u^2} = 2U$, $v^2 + \frac{1}{v^2} = 2V$,

$\sqrt{U^2+V^2-1} = W$ and $2S = U+V+W+1$, then

$F\ (\sqrt{S} - \sqrt{S-1})^2(\sqrt{S-U} - \sqrt{S-U-1})^2(\sqrt{S-V} - \sqrt{S-V-1})^2(\sqrt{S-W} - \sqrt{S-W-1})^2$

$e^{-\pi\sqrt{n}}$.

$\sqrt{210}$:- $(4-\sqrt{15})^4(\sqrt{10}-3)^4(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(6-\sqrt{35})^2$

$\times(\sqrt{15}-\sqrt{14})^2(3-2\sqrt{2})^2(2-\sqrt{3})^2$:

If $2u = 11 + \frac{f^6(-x)}{x f^6(-x^5)}$ and $2v = 1 + \frac{f(-x^{1/5})}{x^{1/5} f(-x^5)}$.

then $\sqrt[5]{\sqrt{u^2+1} - u} = \sqrt{v^2+1} - v = \frac{\sqrt[5]{x}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+\&c}$

$$3 + \frac{f^3(-x^{1/3})}{x^{1/3} f^3(-x^3)} = \sqrt[3]{27 + \frac{f^{12}(-x)}{x f^{12}(-x^3)}}$$

$$f^3(-x^{1/3}) + 3x^{1/3} f^3(-x^3) = f(-x)\left\{1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \&c\right)\right\}$$

$$1 + \frac{\psi(-x^{1/3})}{x^{1/3}\psi(-x^3)} = \sqrt[3]{1 + \frac{\psi^4(-x)}{x\psi^4(-x^3)}}$$

$$\frac{\phi(x^{1/3})}{\phi(x^3)} - 1 = \sqrt[3]{\frac{\phi^4(x)}{\phi^4(x^3)} - 1}.$$

$$1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1.1}{2.4}\right)^2 x^2 + \left(\tfrac{1.1.3}{2.4.6}\right)^2 x^3 + \&c$$

$$\doteq \frac{\pi}{3}(1+x) + \frac{2}{3\pi}\left\{1 - 24\left(\frac{1}{e^{2\pi}-1} + \frac{2}{e^{4\pi}-1} + \&c\right)\right\}$$

$$1 - \frac{1^2}{2^2}x - \frac{1^2.3}{2^2.4^2}x^2 - \frac{1^2.3^2.5}{2^2.4^2.6^2}x^3 - \&c$$

$$= \frac{\pi}{3}(2-x) + \frac{1}{3\pi}\left\{1 - 24\left(\frac{1}{e^{2\pi}-1} + \frac{2}{e^{4\pi}-1} + \&c\right)\right\}$$

$$\frac{\cos\theta + 2\cos\frac{\theta}{2}\cosh\frac{\theta}{2}\sqrt{3}}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{\cos 3\theta + 2\cos\frac{3\theta}{2}\cosh\frac{3\theta\sqrt{3}}{2}}{3\cosh\frac{3\pi\sqrt{3}}{2}} + \&c = \frac{\pi}{8}.$$

$$\frac{1^5}{1^6 - x^6}\,\frac{1}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{3^5}{3^6 - x^6}\,\frac{1}{\cosh\frac{3\pi\sqrt{3}}{2}} + \&c$$

$$= \frac{\pi}{12}\cdot\frac{1}{\cos\frac{\pi x}{2}}\left\{\cos\frac{\pi x}{2} + \cosh\frac{\pi x\sqrt{3}}{2}\right\}$$

If $u = \frac{f^4(-x)}{x f^4(-x^7)}$ and $v = \frac{f(-x^{1/7})}{x^{2/7} f(-x^7)}$, then

$$2u = 7(v^3+5v^2+7v) + (v^2+7v+7)\sqrt{4v^3+21v^2+28v}$$

$$1+12\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\&c\right)-12\left(\frac{3x^3}{1-x^3}+\frac{6x^6}{1-x^6}+\&c\right)$$
$$= \frac{\{\psi^4(x)+3x\psi^4(x^3)\}^2}{\psi^2(x)\psi^2(x^3)} = \frac{\{f^{12}(-x)+27x f^{12}(-x^3)\}^{\frac{2}{3}}}{f^4(-x) f^4(-x^3)}$$

$$= \left\{\frac{\phi^4(\sqrt{x})+3\phi^4(\sqrt{x^3})}{4\phi(\sqrt{x})\phi(\sqrt{x^3})}\right\}^2$$

$$1+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-12\left(\frac{3x^6}{1-x^6}+\frac{6x^{12}}{1-x^{12}}+\&c\right)$$
$$= \left\{\frac{\phi^4(x)+3\phi^4(x^3)}{4\phi(x)\phi(x^3)}\right\}^2 = \phi^2(x)\phi^2(x^3)\left\{1-\frac{4x}{\chi^6(x)\chi^6(x^3)}\right\}$$

$$1+6\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\&c\right)-6\left(\frac{5x^5}{1-x^5}+\frac{10x^{10}}{1-x^{10}}+\&c\right)$$
$$= \sqrt{f^{12}(-x)+22x f^6(-x) f^6(-x^5)+125x^2 f^{12}(-x^5)} \Big/ f(-x)f(-x^5).$$

$$= \{\psi^4(x)+2x\psi^2(x)\psi^2(x^5)+5x^2\psi^4(x^5)\}\sqrt{\psi^4(x)-2x\psi^2(x)\psi^2(x^5)+5x^2\psi^4(x^5)}$$
$$\div\ \psi(x)\psi(x^5).$$

$$1+6\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-6\left(\frac{5x^{10}}{1-x^{10}}+\frac{10x^{20}}{1-x^{20}}+\&c\right)$$
$$= \phi^2(x)\phi^2(x^5)\left\{1-\frac{2x}{\chi^4(x)\chi^4(x^5)}\right\}\sqrt{1-\frac{4x}{\chi^4(x)\chi^4(x^5)}}$$

$$1+4\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)-4\left(\frac{7x^7}{1-x^7}+\frac{14x^{14}}{1-x^{14}}+\&c\right)$$
$$= \left\{\frac{f^8(-x)+13xf^4(-x)f^4(-x^7)+49x^2f^8(-x^7)}{f(-x)f(-x^7)}\right\}^{\frac{2}{3}}$$

$$1+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\right)-4\left(\frac{7x^{14}}{1-x^{14}}+\frac{14x^{28}}{1-x^{28}}+\&c\right)$$

$$=\phi^2(x)\phi^2(x^7)\left\{1-\frac{2x}{\chi^3(x)\chi^3(x^7)}\right\}^2.$$

$$1+3\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\&c\right)-3\left(\frac{9x^9}{1-x^9}+\frac{18x^{18}}{1-x^{18}}+\&c\right)$$

$$=\frac{f^6(-x^3)}{f^2(-x)f^2(-x^9)}\left\{f^6(-x)+9xf^3(-x)f^3(-x^9)+27x^2f^6(-x^9)\right\}^{\frac{1}{3}}.$$

$$\left\{\frac{\phi^4(x^3)+3\phi^2(x)\phi^2(x^9)}{4}\right\}^2\cdot\frac{\phi^2(x^3)}{\phi^3(x)\phi^3(x^9)}=1+3\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-3\left(\frac{9x^{18}}{1-x^{18}}+\&c\right)$$

$$*\quad \overset{u}{\frac{f(-x^2,-x^7)}{f(-x,-x^8)}}=\overset{v}{\frac{f(-x^4,-x^5)}{f(-x^2,-x^7)}}+x\,\overset{w}{\frac{f(-x,-x^8)}{f(-x^4,-x^5)}}$$

$$f(-x^{\frac13})=f(-x^4,-x^5)-x^{\frac13}f(-x^2,-x^7)-x^{\frac23}f(-x,-x^8).$$

$$*\quad \frac{1}{u}+\frac{1}{v}+\frac{1}{w}=\frac{f^4(-x^3)}{f(-x)f(-x^9)}$$

$$5+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\frac{3x^6}{1-x^6}+\&c\right)-12\left(\frac{11x^{22}}{1-x^{22}}+\frac{22x^{44}}{1-x^{44}}+\&c\right)$$

$$=5\phi^2(x)\phi^2(x^{11})-20xf^2(x)f^2(x^{11})+32x^2f^2(-x^2)f^2(-x^{22})$$
$$-20x^3\psi^2(-x)\psi^2(-x^{11}).$$

$$=\phi^2(x)\phi^2(x^{11})\left[5\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-\frac{1}{2}\left\{1-\sqrt[4]{\alpha\beta}-\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^2\right]$$

$$3+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-4\left(\frac{19x^{38}}{1-x^{38}}+\frac{38x^{76}}{1-x^{76}}+\&c\right)$$

$$=\phi^2(x)\phi^2(x^{19})\left[3\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-\frac{1}{2}\left\{1-\sqrt[4]{\alpha\beta}-\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^2\right]$$

$$11+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-12\left(\frac{23x^{46}}{1-x^{46}}+\frac{46x^{92}}{1-x^{92}}+\&c\right)$$

$$=\phi^2(x)\phi^2(x^{23})\left\{11\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\right.$$
$$-16\sqrt[3]{2}\cdot\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\cdot\frac{1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}}{2}$$
$$\left.-10\sqrt[3]{4}\cdot\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\right\}.$$

$$7 + 12\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 12\left(\frac{15x^{30}}{1-x^{30}} + \frac{30x^{60}}{1-x^{60}} + \&c\right)$$

$$= \phi^2(x)\,\phi^2(x^{15})\left\{7\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2} - \frac{3}{2}\cdot\frac{4}{\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}}\left\{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right\}\right.$$

$$? = \frac{\lambda}{\lambda+}\,\frac{\lambda+1}{\lambda+1+}\,\frac{\lambda+2}{\lambda+2} \qquad - \frac{8\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}{\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)^2}\Bigg\}$$

$$5 + 4\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 4\left(\frac{31x^{62}}{1-x^{62}} + \frac{62x^{124}}{1-x^{124}} + \&c\right)$$

$$= \phi^2(x)\,\phi^2(x^{31})\left\{5\cdot\frac{1+\sqrt{\alpha\beta}\pm\sqrt{(1-\alpha)(1-\beta)}}{2} - 6\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)\right.$$

$$\left. - 4\sqrt[16]{\alpha\beta(1-\alpha)(1-\beta)}\sqrt{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}}\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)\right\}$$

$$1 + 6\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 6\left(\frac{5x^{10}}{1-x^{10}} + \frac{10x^{20}}{1-x^{20}} + \&c\right)$$

$$= \phi^2(x)\,\phi^2(x^5)\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}\left\{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2} + \frac{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}}{4}\right.$$

~~$= \phi^2(x)\,\phi^2(x^5)\sqrt{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}$~~

$$= \phi^2(x)\,\phi^2(x^5)\sqrt{\frac{1+\alpha\beta+(1-\alpha)(1-\beta)}{2}} - \frac{3}{\sqrt[3]{4}}\sqrt[3]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$v = \frac{\sqrt[3]{x}}{1+}\,\frac{x+x^2}{1+}\,\frac{x^2+x^4}{1+}\,\frac{x^3+x^6}{1+\&c} = \sqrt[3]{x}\,\frac{\chi(-x)}{\chi^3(-x^3)}$$

$$\frac{1}{v} + 1 = \frac{\psi(x^{\frac{1}{3}})}{x^{\frac{1}{3}}\,\psi(x^3)}; \qquad \frac{1}{v^3} + 1 = \frac{\psi^4(x)}{x\,\psi^4(x^3)}.$$

$$= \sqrt{x}\,\frac{f(-x,-x^7)}{f(-x^3,-x^5)}$$

$$v = \frac{\sqrt{x}}{1+x+}\,\frac{x^2}{1+x^3+}\,\frac{x^4}{1+x^5+}\,\frac{x^6}{1+x^7+\&c}$$

$$\frac{1}{v} - v = \frac{\phi(x^2)}{\sqrt{x}\,\psi(x^4)}; \qquad \frac{1}{v} + v = \frac{\phi(x)}{\sqrt{x}\,\psi(x^4)}$$

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$$\left(\sqrt{\frac{46+7\sqrt{43}}{4}}-\sqrt{\frac{42+7\sqrt{43}}{4}}\right)^{12}.$$

$$f(-x^7,-x^8)+xf(-x^2,-x^{13})=\frac{f(-x^3,-x^5)}{f(-x,-x^4)}f(-x^5).$$

$$f(-x^4,-x^{11})-xf(-x,-x^{14})=\frac{f(-x,-x^4)}{f(-x^2,-x^3)}f(-x^5).$$

$$f(-x,-x^{14})f(-x^4,-x^{11})f(-x^6,-x^9)f(-x^5)$$
$$=f(-x^2,-x^3)f^3(-x^{15}).$$
$$f(-x^7,-x^8)f(-x^2,-x^{13})f(-x^3,-x^{12})f(-x^5)$$
$$=f(-x^2,-x^3)f^3(-x^{15}).$$

$$f(-x^7,-x^8)-xf(-x^2,-x^{13})=f(-x^{\frac{2}{3}},-x)+x^{\frac{1}{3}}f(-x^3,-x^{12})$$

$$f(-x^4,-x^{11})+xf(-x,-x^{14})=\frac{f(-x^6,-x^9)-f(-x^{\frac{1}{3}},-x^{\frac{4}{3}})}{x^{\frac{1}{3}}}.$$

$$\frac{\sqrt{2}}{1}+\frac{x}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+\frac{x^4}{1+x^4}+\&c=8x\frac{\psi(x)}{\phi(x)}.$$

$$\frac{f(-x^{\frac{1}{7}})}{f(-x^7)}=\frac{f(-x^2,-x^5)}{f(-x,-x^6)}-x^{\frac{1}{7}}\frac{f(-x^3,-x^4)}{f(-x^2,-x^5)}-x^{\frac{2}{7}}$$
$$+x^{\frac{5}{7}}\frac{f(-x,-x^6)}{f(-x^3,-x^4)}.$$

$$\frac{f(-x^{\frac{1}{11}})}{f(-x^{11})}=\frac{f(-x^4,-x^7)}{f(-x^2,-x^9)}-x^{\frac{1}{11}}\frac{f(-x^3,-x^8)}{f(-x,-x^{10})}-x^{\frac{2}{11}}\frac{f(-x^5,-x^6)}{f(-x^3,-x^8)}$$
$$+x^{\frac{5}{11}}+x^{\frac{7}{11}}\frac{f(-x^2,-x^9)}{f(-x^5,-x^6)}-x^{\frac{15}{11}}\frac{f(-x,-x^{10})}{f(-x^4,-x^7)};$$

$f(-x^{\frac{1}{7}}) + x^{?} f(-x^7) = \sqrt[7]{u} - \sqrt[7]{v} + \sqrt[7]{w}$

$\sqrt[7]{uvw} = x^{\frac{4}{7}} f^{8}(-x^7)$.

$$u - v + w = \frac{f^8(-x) + 14x f^4(-x) f^4(-x^7) + 57x^2 f^8(-x^7)}{f(-x^7)}$$

$$uv - uw + vw = f^2(-x^7)\{f^{12}(-x) + 19x f^8(-x) f^4(-x^7)$$

$$+ 126x^2 f^4(-x) f^8(-x^7) + 289x^3 \ldots$$

$$\frac{f(-x^{\frac{1}{13}})}{x^{\frac{7}{13}} f(-x^{13})} = \frac{f(-x^4, -x^9)}{x^{\frac{2}{13}} f(-x^2, -x^{11})} - \frac{f(-x^6, -x^7)}{x^{[illegible]} f(-x^3, -x^{10})} - \frac{f(-x^2, -x^{11})}{x^{\frac{1}{13}} f(-x, \ldots)}$$

$$+ \frac{f(-x^5, -x^8)}{x^{\frac{1}{13}} f(-x^4, -x^9)} + 1 - x^{\frac{5}{13}} \frac{f(-x^3, -x^{10})}{f(-x^5, -x^8)}$$

$$+ x^{\frac{15}{13}} \frac{f(-x, -x^{12})}{f(-x^6, -x^7)}$$

$$= u_1 - u_2 - u_3 + u_4 + 1 - u_5 + u_6$$

$$u_1 u_2 - u_3 u_5 - u_4 u_6 = 1 + \frac{f^2(-x)}{x f^2(-x^{13})}.$$

$$u_1 u_2 \cdot u_3 u_5 - u_3 u_5 \cdot u_4 u_6 + u_4 u_6 \cdot u_1 u_2 = 4 + \frac{f^2(-x)}{x f^2(-x^{13})}$$

$$u_1 u_2 u_3 u_4 u_5 u_6 = 1.$$

$$u_2 u_3 u_4 - u_1 u_5 u_6 = 3 + \frac{f^2(-x)}{x f^2(-x^{13})}$$

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$$\frac{f(-x^{\frac{1}{17}})}{x^{\frac{12}{17}} f(-x^{17})} = \frac{f(-x^6,-x^{11})}{x^{\frac{12}{17}} f(-x^3,-x^{14})} - \frac{1}{x^{\frac{11}{17}}} \frac{f(-x^4,-x^{13})}{f(-x^2,-x^{15})}$$

$$- \frac{1}{x^{\frac{10}{17}}} \cdot \frac{f(-x^8,-x^9)}{f(-x^4,-x^{13})} + \frac{1}{x^{\frac{7}{17}}} \frac{f(-x^2,-x^{15})}{f(-x,-x^{16})}$$

$$+ \frac{1}{x^{\frac{3}{17}}} \frac{f(-x^7,-x^{10})}{f(-x^3,-x^{14})} - 1 - x^{\frac{5}{17}} \frac{f(-x^5,-x^{12})}{f(-x^6,-x^{11})}$$

$$+ x^{\frac{14}{17}} \frac{f(-x^3,-x^{14})}{f(-x^7,-x^{10})} - x^{\frac{28}{17}} \frac{f(-x,-x^{16})}{f(-x^8,-x^9)}$$

$$= u_1 - u_2 - u_3 + u_4 + u_5 - 1 - u_6 + u_7 - u_8.$$

$$u_1 u_5 + u_2 u_8 - u_3 u_4 - u_6 u_7 = -1$$

$$u_1 u_5 \cdot u_6 u_7 = 1$$

$$u_2 u_8 \cdot u_3 u_4 = 1$$

1, 3, 9, 27 $\qquad \dfrac{\phi(x^3)\phi(x^9)}{\phi(x)\phi(x^{27})} = \dfrac{\sqrt[4]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)}}$

1, 5, 7, 35 $\qquad \dfrac{\phi(x^5)\phi(x^7)}{\phi(x)\phi(x^{35})} = \dfrac{\sqrt[4]{16\beta\gamma(1-\beta)(1-\gamma)} - \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)}}$

$$= \frac{\sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)} + \sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)} - \sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}$$

$$\{1 - \sqrt[3]{\alpha\beta} - \sqrt[3]{(1-\alpha)(1-\beta)}\}^3.$$

$$1 - \sqrt[3]{\alpha\beta} - \sqrt[3]{(1-\alpha)(1-\beta)} = P \qquad P + P_1 =$$

$$9\sqrt[3]{\alpha\beta(1-\alpha)(1-\beta)} = R \qquad P_1 = \sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)}$$

$$8.\ P^3 + 36PR - \frac{162R^2}{P} = 81R.$$

$$8.\ P^3 + 4PR - \frac{2R^2}{P} = 9R.$$

$$P^3 - (5P + 9P_1)R - \frac{2R^2}{P} = 0$$

$$\frac{1}{2n} + \frac{n^2}{1(1^2+n^2)} + \frac{n^2}{2(2^2+n^2)} + \frac{n^2}{3(3^2+n^2)} + \&c.$$

$$+ \frac{4n^2}{1^4-n^4}\cdot\frac{1}{e^{2\pi}-1} + \frac{8n^2}{2^4-n^4}\cdot\frac{1}{e^{4\pi}-1} + \frac{12n^2}{3^4-n^4}\cdot\frac{1}{e^{6\pi}-1} + \&c$$

$$= \frac{1}{2\pi n^2} - \frac{\pi\cot\pi n}{e^{2\pi n}-1} + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

$$\frac{n^2}{1(1^2+n^2)} + \frac{n^2}{3(3^2+n^2)} + \frac{n^2}{5(5^2+n^2)} + \&c$$

$$+ \left(\frac{4n^2}{1^4-n^4}\cdot\frac{1}{e^{\pi}+1} + \frac{12n^2}{3^4-n^4}\cdot\frac{1}{e^{3\pi}+1} + \&c\right)$$

$$= -\frac{\pi}{2}\cdot\frac{\tan\frac{\pi n}{2}}{e^{\pi n}+1} + 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n-1}$$

$$\frac{1}{2n} + 2n\left\{\frac{1}{(1^2-n^2)}\cdot\frac{1}{e^{\pi}-1} - \frac{1}{3^2-n^2}\cdot\frac{1}{e^{3\pi}-1} + \&c\right\}$$

$$+ 2n\left\{\frac{1}{(2^2+n^2)}\cdot\frac{1}{e^{\pi}+e^{-\pi}} + \frac{1}{4^2+n^2}\cdot\frac{1}{e^{2\pi}+e^{-2\pi}} + \&c\right\}$$

$$= \frac{\pi}{2}\cdot\frac{\sec\frac{\pi n}{2}}{e^{\pi n}-1} + \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+5} - \frac{1}{n+7} + \&c.$$

(1) $P = \frac{f(-x)}{x^{1/12} f(-x^3)}$ & $Q = \frac{f(-x^5)}{x^{5/12} f(-x^{15})}$ · $(PQ)^2 + \left(\frac{3}{PQ}\right)^2 + 5 = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3$.

(2) If $P = \frac{f(-x)}{x^{1/12} f(-x^3)}$ & $Q = \frac{f(-x^7)}{x^{7/12} f(-x^{21})}$, then

$$(PQ)^3 + \left(\frac{3}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^4 - 7\cdot\left(\frac{Q}{P}\right)^2 + 7\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^4.$$

(4) If $P = \frac{f(-x^3)}{x^{2} f(-x^5)}$ & $Q = \frac{f(-x^3)}{x^{2} f(-x^{15})}$

$$(PQ)^3 + \left(\frac{5}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^6 - 9\cdot\left(\frac{Q}{P}\right)^3 - 9\cdot\left(\frac{P}{Q}\right)^3 - \left(\frac{P}{Q}\right)^6.$$

$$\frac{1}{2x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \&c$$

$$= \frac{1}{x} + \frac{1}{2x^2} \cdot \frac{1}{3x +}\ \frac{3}{5x +}\ \frac{18}{7x +}\ \frac{60}{9x + \&c}$$

$3 = 2^2(2^2-1)/4$

$18 = 3^2(3^2-1)/4$

$60 = 4^2(4^2-1)/4$ &c.

$$\frac{1}{2x^3} + \frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&c$$

$$= \frac{1}{2x^2} + \frac{1}{4x^3} \cdot \frac{1}{x +}\ \frac{1}{3x +}\ \frac{2}{x +}\ \frac{6}{5x +}\ \frac{9}{x +}\ \frac{18}{7x + \&c}$$

$$\frac{x}{1+n+}\;\frac{1^2x^2}{3+n+}\;\frac{2^2x^2}{5+n+}\;\frac{3^2x^2}{7+n+\&c}$$

$$= 2\left(\frac{y}{m+1} - \frac{y^3}{m+3} + \frac{y^5}{m+5} - \&c\right)$$

&

$$\frac{x}{2+n+}\;\frac{1.2x^2}{4+n+}\;\frac{2.3x^2}{6+n+}\;\frac{3.4x^2}{8+n+\&c}$$

$$= y - m\left(y+\frac{1}{y}\right)\left(\frac{y^2}{m+2} - \frac{y^4}{m+4} + \frac{y^6}{m+8} - \&c\right)$$

where $y = \dfrac{\sqrt{1+x^2}-1}{x}$ and $m = \dfrac{n}{\sqrt{1+x^2}}$.

$$2^p\left\{\frac{1}{n+p} - \frac{p}{\lfloor 1}\cdot\frac{1}{n+p+2} + \frac{p(p+1)}{\lfloor 2}\cdot\frac{1}{n+p+4} - \&c\right\}$$

$$= \frac{1}{n+}\;\frac{1.p}{n+}\;\frac{2(p+1)}{n+}\;\frac{3(p+2)}{n+\&c}$$

$$\frac{x}{p+n+}\;\frac{1.px^2}{p+2+n+}\;\frac{2(p+1)x^2}{p+4+n+}\;\frac{3(p+2)x^2}{p+6+n+\&c}$$

$$=\left(1+\frac{1}{x^2}\right)^{\frac{p-1}{2}}(2y)^p\left\{\frac{1}{m+p} - \frac{p}{\lfloor 1}\cdot\frac{y^2}{m+p+2} + \frac{p(p+1)}{\lfloor 2}\cdot\frac{y^4}{m+p+4} - \&c\right\}$$

where $y = \dfrac{\sqrt{1+x^2}-1}{x}$ and $m = \dfrac{n}{\sqrt{1+x^2}}$

$$\frac{\pi}{2}\int_0^\infty \frac{dx}{e^{x^n}+e^{-x^n}}$$

$$= \sqrt[n]{\frac{\pi}{2}}\,\Gamma\!\left(\frac{1}{n}\right)\cos\frac{\pi}{2n}\int_0^\infty \frac{x^{n-2}\,dx}{e^{x^n}+e^{-x^n}}$$

$$+ \left\lfloor \frac{1}{n}-1 \right.$$

$$\frac{x}{4n+2} + \frac{x^2}{4n+6+} \frac{x^2}{4n+10+} \&c$$

$$+ \frac{2n}{x+} \frac{n-1}{1-} \frac{n+1}{x+} \frac{n-2}{1-} \frac{n+2}{x-} \&c$$

$= 1$ nearly.

$$\frac{1}{x+} \frac{a}{x+} \frac{a^2}{x+} \frac{a^3}{x+} \&c = 1 - \frac{ax}{1+a+} \frac{a^2x}{1+a^2-} \frac{a^3x}{1+a^3+} \frac{a^6x}{1+a^4} - \frac{a^7x}{1+a^5+} \frac{a^9x}{1+a^6-} \&c$$

nearly.

$$\frac{1}{1-} \frac{a}{1+a-} \frac{a^3}{1+a^2-} \frac{a^5}{1+a^3-} \&c$$

$$= \frac{1-a^2}{1-a}\cdot\frac{1-a^5}{1-a^6}\cdot\frac{1-a^8}{1-a^7}\cdot \&c$$

$$\frac{1-a^3}{1-a}\cdot\frac{1-a^7}{1-a^5}\cdot\frac{1-a^{11}}{1-a^9}\,\&c = \frac{1}{1-} \frac{a}{1+a^2-} \frac{a^3}{1+a^4-} \frac{a^5}{1+a^6}$$

$$\frac{1}{1+} \frac{x}{1+} \frac{x^2+x}{1+} \frac{x^3}{1+} \frac{x^4+x^2}{1+} \frac{x^5}{1+}$$

$$= \frac{\psi(x^2)}{\psi(x)}$$

$$\frac{f(-x,-x^7)}{f(-x^3,-x^5)} = \cfrac{1}{1+\cfrac{x+x^2}{1+\cfrac{x^4}{1+\cfrac{x^3+x^6}{1+\cfrac{x^8}{1+\&c}}}}}$$

$$m\left\{\frac{1}{2(m^2+n^2)} + \frac{1}{m^2+(1+n)^2} + \frac{1}{m^2+(2+n)^2} + \&c\right\}$$

$$+n\left\{\frac{1}{2(m^2+n^2)} + \frac{1}{n^2+(1+m)^2} + \frac{1}{n^2+(2+m)^2} + \&c\right\} -$$

$$= \frac{\pi}{2} + \frac{mn}{\pi(m^2+n^2)^2} + 4mn\left\{\frac{1}{e^{2\pi}-1}\left(\frac{1}{m^2+\overline{1+n}^2}\cdot\frac{1}{m^2+\overline{1-n}^2} + \frac{1}{n^2+\overline{1+m}^2}\cdot\frac{1}{n^2+\overline{1-m}^2}\right)\right.$$

$$\left. + \frac{2}{e^{4\pi}-1}\cdot\left(\frac{1}{m^2+\overline{2+n}^2}\cdot\frac{1}{m^2+\overline{2-n}^2} + \frac{1}{n^2+\overline{2+m}^2}\cdot\frac{1}{n^2+\overline{2-m}^2}\right) + \&c\right\}$$

$$-2\pi\cdot\frac{1-e^{2\pi m}\cos 2\pi n - e^{2\pi n}\cos 2\pi m + e^{2\pi(m+n)}\cos 2\pi(m-n)}{(e^{4\pi m}-2e^{2\pi m}\cos 2\pi n+1)(e^{4\pi n}-2e^{2\pi n}\cos 2\pi m+1)}$$

If $$\phi(\alpha,\beta) = \alpha\left\{\frac{1}{2(\alpha^2+\beta^2)} + \frac{1}{\alpha^2+(1+\beta)^2} + \frac{1}{\alpha^2+(2+\beta)^2} + \&c\right\}$$

$$- 4\alpha\beta\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{\alpha^2+\overline{1+\beta}^2}\cdot\frac{1}{\alpha^2+\overline{1-\beta}^2} + \frac{2}{e^{4\pi}-1}\cdot\frac{1}{\alpha^2+\overline{2+\beta}^2}\cdot\frac{1}{\alpha^2+\overline{2-\beta}^2}\right\}$$

$$+ \frac{\pi}{e^{4\pi\alpha} - 2e^{2\pi\alpha}\cos 2\pi\beta + 1}.$$

then $$\phi(\alpha,\beta) + \phi(\beta,\alpha) = \frac{\pi}{2} + \frac{\alpha\beta}{\pi(\alpha^2+\beta^2)^2}$$

$$+ \frac{\pi}{2}\cdot\frac{\cosh 2\pi(\alpha-\beta) - \cos 2\pi(\alpha-\beta)}{(\cosh 2\pi\alpha - \cos 2\pi\beta)(\cosh 2\pi\beta - \cos 2\pi\alpha)}$$

If $$\phi(\alpha,\beta) = \alpha\left\{\frac{1}{\alpha^2+(1+\beta)^2} + \frac{1}{\alpha^2+(3+\beta)^2} + \frac{1}{\alpha^2+(5+\beta)^2} + \&c\right\}$$

$$+ 4\alpha\beta\left\{\frac{1}{e^{\pi}+1}\cdot\frac{1}{\alpha^2+\overline{1+\beta}^2}\cdot\frac{1}{\alpha^2+\overline{1-\beta}^2} + \&c\right\}$$

$$+ \frac{\pi/2}{e^{2\pi\alpha} + 2e^{\pi\alpha}\cos\pi\beta + 1},$$ then

$$\phi(\alpha,\beta) + \phi(\beta,\alpha) = \frac{\pi}{4} + \frac{\pi}{4}\cdot\frac{\cosh\pi(\alpha-\beta) - \cos\pi(\alpha-\beta)}{(\cosh\pi\alpha + \cos\pi\beta)(\cosh\pi\beta + \cos\pi\alpha)}$$

$$\frac{1}{4} + \frac{1}{1+(2n)^2} \cdot \frac{1}{e^{\pi m}+e^{-\pi m}} + \frac{1}{1+(4n)^2} \cdot \frac{1}{e^{2\pi m}+e^{-2\pi m}}$$

$$+ m\left\{\frac{1}{n^2-m^2} \cdot \frac{1}{e^{\frac{\pi}{m}}-1} - \frac{1}{(3n)^2-m^2} \cdot \frac{1}{e^{\frac{2\pi}{m}}-1} + \&c\right\}$$

$$= \frac{\pi}{4n} \cdot \frac{\sec \frac{\pi m}{2n}}{e^{\frac{\pi}{n}}-1} + \frac{1}{2}\left(\frac{1}{n+m} - \frac{1}{3n+m} + \frac{1}{5n+m} - \&c\right)$$

$$\frac{x}{8\pi} + \frac{\sin x}{1(e^{2\pi}-1)} + \frac{\sin 4x}{2(e^{4\pi}-1)} + \frac{\sin 9x}{3(e^{6\pi}-1)} + \cdots$$

$$= \frac{1}{4}\left\{\frac{B_2}{1\lfloor 1}x - \frac{B_6}{3\lfloor 3}x^3 + \frac{B_{10}}{5\lfloor 5}x^5 - \&c\right\}$$

If $\alpha\beta = \frac{\pi^2}{2}$

$$\alpha^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\alpha + \cos\alpha} - \frac{3^3 \operatorname{sech}\frac{3\pi}{2}}{\cosh 3\alpha + \cos 3\alpha} + \&c\right\}$$

$$= \beta^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\beta + \cos\beta} - \frac{3^3 \operatorname{sech}\frac{3\pi}{2}}{\cosh 3\beta + \cos 3\beta} + \&c\right\}$$

$$\int_0^\infty \frac{\sin 2nx}{x(\cosh \pi x + \cos \pi x)}$$

$$= \frac{\pi}{4} - 2\left\{\frac{e^{-n}\cos n}{\cosh\frac{\pi}{2}} - \frac{e^{-3n}\cos 3n}{3\cosh\frac{3\pi}{2}} + \&c\right\}$$

If $\alpha\beta = \frac{\pi^2}{4}$, then

$$\frac{1}{\cosh\alpha + \cos\alpha} - \frac{1}{3(\cosh 3\alpha + \cos 3\alpha)} + \&c$$

$$+ \frac{2\cos\beta\cosh\beta}{\cosh\frac{\pi}{2}\cdot(\cosh 2\beta + \cos 2\beta)} - \frac{2\cos 3\beta\cosh 3\beta}{3\cosh\frac{3\pi}{2}(\cosh 6\beta + \cos 6\beta)}$$

$$+\frac{2\cos 5\beta\cosh 5\beta}{5\cosh^5\frac{\pi}{2}(\cosh 10\beta+\cos 10\beta)} - \&c = \frac{\pi}{8}.$$

If $\alpha\beta = \frac{\pi^2}{2}$, then

$$\frac{\cos\alpha}{\cosh\alpha-\cos\alpha} - \frac{\cos 3\alpha}{3(\cosh 3\alpha-\cos 3\alpha)} + \&c$$

$$= \frac{\pi^3}{32\alpha^2} - \frac{\pi}{8} + \frac{\sin\beta\sinh\beta}{\cosh 2\beta+\cos 2\beta}\cdot\frac{\coth\pi}{1}$$

$$+ \frac{\sin 2\beta\sinh 2\beta}{\cosh 4\beta+\cos 4\beta}\cdot\frac{\coth 2\pi}{2} + \&c$$

$$\frac{B_2}{1.2.2n} + \frac{B_4}{3.4.2^2n^3} - \frac{B_6}{5.6.2^3n^5} - \frac{B_8}{7.8.2^4n^7} + \&c$$

~~$= n + \frac{\pi}{4}n + \log|\underline{n}$~~

$$= \log\frac{e^n\,|\underline{n}}{n^n\sqrt{2\pi n}} + n\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$$

$$-\frac{1}{2}\log\left[\sqrt{2}\left\{1+\left(\frac{n}{n+1}\right)^2\right\}\left\{1+\left(\frac{n}{n+2}\right)^2\right\}\&c\right].$$

$$\left\{1+\left(\frac{n}{1}\right)^3\right\}\left\{1+\left(\frac{n}{2}\right)^3\right\}\left\{1+\left(\frac{n}{3}\right)^3\right\}\&c$$

$$\times\left\{1+3\cdot\left(\frac{n}{n+2}\right)^2\right\}\left\{1+3\cdot\left(\frac{n}{n+4}\right)^2\right\}\left\{1+3\cdot\left(\frac{n}{n+6}\right)^2\right\}\&c$$

$$= \frac{|\underline{\tfrac{n}{2}-1}}{|\underline{\tfrac{n-1}{2}}}\cdot\frac{\cosh\pi n\sqrt{3} - \cos\pi n}{2^{n+2}\cdot\pi n\sqrt{\pi}}.$$

$$\frac{1}{2}\log(2\pi x)+\frac{1}{3}\log\left(1+\frac{x^3}{1^3}\right)\left(1+\frac{x^3}{2^3}\right)\left(1+\frac{x^3}{3^3}\right)\ \&c$$

$$=\frac{2\pi x}{3\sqrt{3}}+\frac{B_4}{3.4x^3}-\frac{B_{12}}{9.10x^9}+\frac{B_{16}}{15.16.x^{15}}\ \&c$$

$$\frac{1}{2}\log(2\pi n)+\frac{1}{3}\log\left(1+\frac{n^3}{1^3}\right)\left(1+\frac{n^3}{2^3}\right)\left(1+\frac{n^3}{3^3}\right)\ \&c$$

$$=\frac{1}{3}\log\left(e^{\pi n\sqrt{3}}-2\cos\pi n+e^{-\pi n\sqrt{3}}\right)$$

$$-\frac{\pi n}{3\sqrt{3}}+\frac{B_4}{3.4n^3}-\frac{B_{10}}{9.10n^9}+\ \&c$$

$$\left\{1+\left(\frac{m+n}{1+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{3+m}\right)^3\right\}\ \&c$$

$$\times\left\{1+\left(\frac{m+n}{1+n}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+n}\right)^3\right\}\left\{1+\left(\frac{m+n}{3+n}\right)^3\right\}\ \&c$$

$$=\frac{(\lfloor m)^3(\lfloor n)^3}{\lfloor 2m+n\ \lfloor 2n+m}\cdot\frac{\cosh\pi(m+n)\sqrt{3}-\cos\pi(m-n)}{2\pi^2(m^2+mn+n^2)}$$

$$\int_0^\infty\log(1+x^2)\cos nx\,dx=-\frac{\pi}{n}e^{-n}$$

$$\log\left(1+\frac{x^2}{1^2}\right)-3\log\left(1+\frac{x^2}{3^2}\right)+5\log\left(1+\frac{x^2}{5^2}\right)-\&c$$

$$=\frac{4}{\pi}\cdot\left\{\frac{1-e^{-\frac{\pi x}{2}}}{1^2}-\frac{1-e^{-\frac{3\pi x}{2}}}{3^2}+\frac{1-e^{-\frac{5\pi x}{2}}}{5^2}-\&c\right\}$$

$$-2x\tan^{-1}e^{-\frac{\pi x}{2}}.$$

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$$\log\left(1-\frac{x^2}{1^2}\right) - 3\log\left(1-\frac{x^2}{3^2}\right) + 5\log\left(1-\frac{x^2}{5^2}\right) - \&c$$

$$= \frac{4}{\pi}\left\{\frac{1-\cos\frac{\pi x}{2}}{1^2} - \frac{1-\cos\frac{3\pi x}{2}}{3^2} + \&c\right\}$$

$$+ ~~x~~ \log ~~\tan~~ \frac{\pi - \pi x}{4}$$

$$= \frac{4}{\pi}\left\{\frac{1-\tan\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)}{1^2} - \frac{1-\tan^3\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)}{3^2} + \&c\right\}$$

$$+ \log\tan\frac{\pi - \pi x}{4}.$$

If $\frac{\pi\alpha}{2} = \log\tan\left(\frac{\pi}{4}+\frac{\pi\beta}{4}\right)$

then $\log\left(1+\frac{\alpha^2}{1^2}\right) - 3\log\left(1+\frac{\alpha^2}{3^2}\right) + 5\log\left(1+\frac{\alpha^2}{5^2}\right) - \&c$

$$= \frac{\pi\alpha\beta}{2} + \log\left(1-\frac{\beta^2}{1^2}\right) - 3\log\left(1-\frac{\beta^2}{3^2}\right) + 5\log\left(1-\frac{\beta^2}{5^2}\right) - \&c$$

$$\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\cdot\frac{dx}{n^2+x^2} = \int_0^1 \frac{x^n}{n}\cdot\frac{\sin a\,dx}{1+2x\cos a+x^2}$$

$$\int_0^\infty \frac{\sin ax}{\sinh\frac{\pi x}{2}}\cdot\frac{dx}{1+x^2} = ~~\frac{\pi}{2}\sinh a~~$$

$$\frac{\phi^2(e^{-\pi\sqrt{\frac{n}{3}}})}{\phi^2(e^{-\pi\sqrt{3n}})} = \sqrt{3}\, a_n.$$

$$a_1 = 1;\ a_5 = \frac{\sqrt{5}-1}{2};\ a_7 = \frac{\sqrt{3}+1}{\sqrt{2}}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{\frac{3}{2}}.$$

$$\frac{\phi^4(e^{-\pi\sqrt{\frac{n}{3}}})}{\phi^4(e^{-\pi\sqrt{3n}})} = 3a_n \cdot \psi(e^{-\sqrt{\pi}}$$

$$a_1 = 1;\ a_5 = \left(\frac{\sqrt{5}-1}{2}\right)^2;\ a_7 = (2+\sqrt{3})\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3;$$

$$a_{11} = (10-3\sqrt{11})(2\sqrt{3}+\sqrt{11});\ a_{13} =$$

$$n\, e^{-\frac{\pi}{4}(n-1)\sqrt{\frac{m}{n}}}\ \frac{\psi^2(e^{-\pi\sqrt{mn}})\,\phi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\,\phi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}$$

$$= a_{m,n}.$$

$$a_{3,5} = \frac{3-\sqrt{5}}{2};\ a_{3,7} = 2-\sqrt{3};\ a_{3,11} = 2\sqrt{3}-\sqrt{11},$$

$$a_{3,13} = \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}}\right)^8;\ a_{3,19} = 2\sqrt{19}-5\sqrt{3}.$$

$$a_{3,23} = \left(\sqrt{\frac{7+4\sqrt{3}}{2}} - \sqrt{\frac{5+4\sqrt{3}}{2}}\right)^2;\ a_{3,31} = (2-\sqrt{3})^3.$$

$$a_{3,59} = 102\sqrt{3} - 23\sqrt{59};\ a_{3,71} = \left(\sqrt{\frac{175+100\sqrt{3}}{2}} - \sqrt{\frac{173+100\sqrt{3}}{2}}\right)^2$$

$$a_{5,9} = (2-\sqrt{3})^2;\ a_{5,11} = \left(\sqrt{\frac{7+\sqrt{5}}{8}} - \sqrt{\frac{\sqrt{5}-1}{8}}\right)^8;$$

$$a_{5,13} = \left(\sqrt{\frac{9+\sqrt{65}}{2}} - \sqrt{\frac{7+\sqrt{65}}{2}}\right)^2;\ a_{5,17} = (\sqrt{17}-4)^2;$$

$$a_{5,29} = \left(\sqrt{49+4\sqrt{145}} - \sqrt{48+4\sqrt{145}}\right)^2;\ \&$$

$$a_{7,9} = \left(\sqrt{\frac{5+\sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21}-3}{8}}\right)^8$$

$a_{3,3} = \frac{1}{\sqrt{3}}$; $a_{3,9} = \frac{1}{(\sqrt[3]{2}+1)^2}$; $a_{3,15} = \frac{2-\sqrt{3}}{3}$

~~$a_{3,}$~~

$$\frac{\phi^2(e^{-\pi})}{\phi^2(e^{-n\pi})} = n\, b_n \sqrt[8]{4\beta(1-\beta)}$$

$b_1 = 1$; $b_3 = \frac{1}{\sqrt[4]{3}}$; $b_5 = 1$; $b_7 = \frac{\sqrt{\frac{3+\sqrt{7}}{\sqrt{2}}}}{\sqrt[8]{7}}$.

$b_{17} =$

$$\frac{1^2 x}{1+x} - \frac{2^2 x^3}{1+x^2} + \frac{3^2 x^6}{1+x^3} - \frac{4^2 x^{10}}{1+x^4} + \&c$$

$$= \phi^2(-x)\left\{x \cdot \frac{1+x}{(1-x)^2} + x^6 \cdot \frac{1+x^3}{(1-x^3)^2} + x^{10}\frac{1+x^5}{(1-x^5)^2} + \&c\right\}$$

$$\frac{1+x}{1-x} - 3^2 x^2 \cdot \frac{1+x^3}{1-x^3} + 5^2 x^6 \cdot \frac{1+x^5}{1-x^5} - \&c$$

$$= \psi^2(x)\left\{1 - \frac{8x^2}{(1+x)^2} + \frac{8x^6}{(1+x^2)^2} - \frac{8x^{12}}{(1+x^3)^2} + \&c\right\}$$

$$x\psi(x)\psi(x^4) = \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^6}{1-x^5} - \frac{x^{10}}{1-x^7} + \&c$$

$$\frac{x^n}{\lfloor n}\left\{1 + \frac{x^2}{\lfloor 1} \cdot \frac{1}{1+n} + \frac{x^4}{\lfloor 2} \cdot \frac{1}{(1+n)(2+n)} + \&c\right\}$$

$$- \frac{x^{-n}}{\lfloor -n}\left\{1 + \frac{x^2}{\lfloor 1} \cdot \frac{1}{1-n} + \frac{x^4}{\lfloor 2} \cdot \frac{1}{(1-n)(2-n)} + \&c\right\}$$

If n is a positive integer

$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2}+\frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2}+\&c=\frac{n}{\pi}\left(\frac{B_{4n}}{8n}+\frac{1^{4n-1}}{e^{2\pi}-1}+\frac{2^{4n-1}}{e^{4\pi}-1}+\&c\right)$$

$$\frac{1}{2x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$$

$$=\frac{1}{2\pi x^3}+\frac{\pi}{3x}-\frac{\pi^2}{\sin^2\pi x\,(e^{2\pi x}-1)}$$

$$+4x\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{(1^4-x^4)^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{1}{(2^4-x^4)^2}+\&c\right\}$$

$$+8\pi x^3\left\{\frac{1}{(e^{\pi}-e^{-\pi})^2}\cdot\frac{1}{1^4-x^4}+\frac{1}{(e^{2\pi}-e^{-2\pi})^2}\cdot\frac{1}{2^4-x^4}+\&c\right\}$$

If $\theta\frac{u}{\sqrt{2}}=v+\frac{1}{2}\cdot\frac{v^5}{5}+\frac{1.3}{2.4}\cdot\frac{v^9}{9}+\&c$, then

$$\frac{u^2}{2\pi^2}=\frac{1}{\sin^2\theta}-\frac{1}{\pi}-8\left(\frac{\cos 2\theta}{e^{2\pi}-1}+\frac{2\cos 4\theta}{e^{4\pi}-1}+\&c\right).$$

$$\cot\theta+\frac{\theta}{\pi}+4\left(\frac{\sin 2\theta}{e^{2\pi}-1}+\frac{\sin 4\theta}{e^{4\pi}-1}+\&c\right)$$

$$=\frac{u}{\sqrt{2}}\left\{\frac{1}{v}-\frac{1}{2}\cdot\frac{v^3}{3}-\frac{1.3}{2.4}\cdot\frac{v^7}{7}-\&c\right\}$$

$$\log\sin\theta+\frac{\theta^2}{2\pi}-2\left\{\frac{\cos 2\theta}{1(e^{2\pi}-1)}+\frac{\cos 4\theta}{2(e^{4\pi}-1)}+\&c\right\}$$

$$=\log\frac{v\sqrt{2}}{u}+\frac{1}{3}\cdot\frac{v^4}{4}+\frac{1.5}{3.7}\cdot\frac{v^8}{8}+\frac{1.5.9}{3.7.11}\cdot\frac{v^{12}}{12}+\&c$$

$$\frac{1}{2}\tan^{-1}v=\frac{\sin\theta}{\cosh\frac{\pi}{2}}+\frac{\sin 3\theta}{3\cosh\frac{3\pi}{2}}+\frac{\sin 5\theta}{5\cosh\frac{5\pi}{2}}+\&c$$

$$\frac{1}{4}\cos^{-1}v^2=\frac{\cos\theta}{\cosh\frac{\pi}{2}}-\frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}}+\frac{\cos 5\theta}{5\cosh\frac{5\pi}{2}}-\&c$$

$$\frac{\pi\theta}{8}-\frac{\sin\theta}{1^2\cosh\frac{\pi}{2}}+\frac{\sin 3\theta}{3^2\cosh\frac{3\pi}{2}}-\&c$$

$$=\frac{\sqrt{2}}{4u}\left\{\frac{v^3}{3}+\frac{2}{3}\cdot\frac{v^7}{7}+\frac{2.4}{3.5}\cdot\frac{v^{11}}{11}+\&c\right\}.$$

If $\theta\frac{u}{2} = v - \frac{1}{2}\cdot\frac{v^5}{5} + \frac{1.3}{2.4}\cdot\frac{v^9}{9} - \&c$

$$2\tan^{-1}v = \theta + \frac{\sin 2\theta}{\cosh\pi} + \frac{\sin 4\theta}{2\cosh 2\pi} + \frac{\sin 6\theta}{3\cosh 3\pi} + \&c$$

$$\frac{\pi}{8} - \frac{1}{2}\tan^{-1}v^2 = \frac{\cos\theta}{\cosh\frac{\pi}{2}} - \frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}} + \frac{\cos 5\theta}{5\cosh\frac{5\pi}{2}} - \&c$$

$$\log\frac{1+v}{1-v} = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + 4\left\{\frac{\sin\theta}{e^{\pi}-1} - \frac{\sin 3\theta}{3(e^{3\pi}-1)} + \&c\right.$$

$$\frac{1}{2\pi x^4} + \coth\pi\left(\frac{1}{1+x^2+x^4} + \frac{1}{1-x^2+x^4}\right)$$
$$+ 2\coth 2\pi\left(\frac{1}{16+4x^2+x^4} + \frac{1}{16-4x^2+x^4}\right)$$
$$+ \&c \qquad \&c$$

$$= \frac{\pi}{x^2\sqrt{3}}\cdot\frac{\sinh\pi x\sqrt{3}\,\sinh\pi x + \sin\pi x\sqrt{3}\,\sin\pi x}{(\cosh\pi x\sqrt{3} - \cos\pi x)(\cosh\pi x - \cos\pi x\sqrt{3})}$$

$$\Sigma\frac{1}{x} - \Sigma\frac{1}{x/3} + \frac{1}{x} - \log 3$$

$$= \frac{2}{3}\cdot\frac{1}{x^2 + \frac{2^3-2}{6 + \frac{4^3-4}{3x^2 + \frac{5^3-5}{6 + \frac{7^3-7}{5x^2 + \&c}}}}}$$

$$\frac{1^{6n}}{\left(e^{\frac{\pi\sqrt{3}}{2}}+e^{-\frac{\pi\sqrt{3}}{2}}\right)^2} - \frac{2^{6n}}{\left(e^{\pi\sqrt{3}}-e^{-\pi\sqrt{3}}\right)^2} + \&c$$

$$+ \frac{n\sqrt{3}}{\pi}\left\{\frac{B_{6n}}{12n}\cos 3\pi n - \left(\frac{1^{6n-1}}{e^{\pi\sqrt{3}}+1} - \&c\right)\right\} = 0$$

n being a positive integer

$$2^2\left\{1+240\left(\frac{1^3}{e^{2\pi\sqrt{2}}-1}+\frac{2^3}{e^{4\pi\sqrt{2}}-1}+\&c\right)\right\}-\left\{1+240\left(\frac{1^3}{e^{\pi\sqrt{2}}-1}+\frac{2^3}{e^{2\pi\sqrt{2}}-1}\right)\right\}=0$$

$$2^3\left\{1-504\left(\frac{1^5}{e^{2\pi\sqrt{2}}-1}+\&c\right)\right\}+\left\{1-504\left(\frac{1^5}{e^{\pi\sqrt{2}}-1}+\&c\right)\right\}=0$$

$$2^4\left\{1+480\left(\frac{1^7}{e^{2\pi\sqrt{2}}-1}+\&c\right)\right\}-\left\{1+480\left(\frac{1^7}{e^{\pi\sqrt{2}}-1}+\&c\right)\right\}=0$$

If $S_n=\frac{B_n}{2n}+\frac{1^{n-1}}{e^{2\pi}-1}+\frac{2^{n-1}}{e^{4\pi}-1}+\&c$

where n is a multiple of 4,

then $$\frac{(n+3)(n-4)}{24}S_{n+2}=\frac{n(n-1)(n-2)(n-3)}{\lfloor 2}S_4S_{n-2}$$

$$+\frac{n(n-1)\ldots(n-7)}{\lfloor 6}S_8S_{n-6}+\&c$$

If $\frac{2}{3}\theta u=v+\frac{1}{2}\cdot\frac{v^7}{7}+\frac{1.3}{2.4}\cdot\frac{v^{13}}{13}+\&c$

then $$\frac{4}{9}\frac{u^2}{v^2}=\frac{1}{\sin^2\theta}-\frac{2}{\pi\sqrt{3}}+8\left(\frac{\cos 2\theta}{e^{\pi\sqrt{3}}+1}-\frac{2\cos 4\theta}{e^{2\pi\sqrt{3}}-1}+\&c\right)$$

where $u=\dfrac{\sqrt{\pi}}{\lfloor -\frac{1}{6}\ \lfloor -\frac{1}{3}}$.

$$\frac{1}{2}\log\left[\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\}\&c\right]$$

$$=\log\lfloor n+n+x\tan^{-1}\frac{x}{n}-\frac{n}{2}\log(n^2+x^2)$$

$$-\frac{1}{2}\log\left(2\pi\sqrt{n^2+x^2}\right)-\int_0^\infty\frac{\tan^{-1}\frac{2nz}{n^2+x^2-z^2}}{e^{2\pi z}-1}\,dz.$$

$\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots\cdots+\sqrt{x}$

$=c+\frac{2}{3}x\sqrt{x}+\frac{1}{2}\sqrt{x}+\frac{1}{6}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^3}+\frac{1}{(\sqrt{x+1}+\sqrt{x+2})^3}+\&c\right\}$

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$1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\cdots\cdots+x\sqrt{x}$

$=c+\frac{2}{5}x^2\sqrt{x}+\frac{x}{2}\sqrt{x}+\frac{\sqrt{x}}{8}+\frac{1}{40}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^5}+\frac{1}{(\sqrt{x+1}+\sqrt{x+2})^5}+\cdots\right\}$

$(1^2\sqrt{1}+2^2\sqrt{2}+3^2\sqrt{3}\quad\cdots\quad+x^2\sqrt{x})+\frac{1}{16}(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{x})$

$=c+\frac{2}{7}x^3\sqrt{x}+\frac{x^2}{2}\sqrt{x}+\frac{x\sqrt{x}}{4}+\frac{\sqrt{x}}{32}+\frac{1}{224}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^7}+\&c\right\}$

~~$1^3\sqrt{1}+2^3\sqrt{2}\cdots+x^3\sqrt{x}+$~~

$c+\frac{\pi}{3}\log x+\frac{1}{2x}-\frac{1}{4\pi x^2}+2\left(\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^2-x^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{1}{2^2-x^2}+\&c\right)$

$+\frac{\pi\cot\pi x}{e^{2\pi x}-1}+\frac{2\pi\log(2\sin\pi x)}{(e^{\pi x}-e^{-\pi x})^2}-2\pi\left\{\frac{\log(1^2-x^2)}{(e^{\pi}-e^{-\pi})^2}+\frac{\log(2^2-x^2)}{(e^{2\pi}-e^{-2\pi})^2}+\&c\right\}$

$-2\pi\sum_{n=1}^{n=\infty}e^{-2\pi nx}\left[n^2\left\{\frac{\sin 2\pi x}{1^2+n^2}+\frac{\sin 4\pi x}{2^2+n^2}+\&c\right\}-n^3\left\{\frac{\cos 2\pi x}{1(1^2+n^2)}+\frac{\cos 4\pi x}{2(2^2+n^2)}+\&c\right\}\right]$

$=1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{x}.$

$y\sqrt[4]{\frac{\sqrt{13}-3}{2}}$ where $\sqrt{5}\left(y^3+y^2\frac{\sqrt{13}-1}{2}+y\cdot\frac{\sqrt{13}+1}{2}-1\right)$

$\pm\left\{y^3+y^2\left(\frac{\sqrt{13}+1}{2}\right)^2+y\left(\frac{\sqrt{13}-1}{2}\right)^2+1\right\}=0$

$\sqrt{765}\quad(\sqrt{5}-2)^8\left(\frac{\sqrt{85}-9}{2}\right)^6(4-\sqrt{15})^6(16-\sqrt{255})^2$

$\times\left(\sqrt{\frac{22+3\sqrt{51}}{4}}-\sqrt{\frac{18+3\sqrt{51}}{4}}\right)^{12}\left(\sqrt{\frac{10+\sqrt{51}}{4}}-\sqrt{\frac{6+\sqrt{51}}{4}}\right)^{12}$

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If $\phi = \cfrac{x}{1+\cfrac{x^5}{1+\cfrac{x^{10}}{1+\ldots}}}$ and $f = \cfrac{x^{1/5}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\ldots}}}}$

then $f^5 = \phi \cdot \dfrac{1-2\phi+4\phi^2-3\phi^3+\phi^4}{1+3\phi+4\phi^2+2\phi^3+\phi^4}$

$\sqrt{505}$ $(\sqrt{5}-2)^{14}(\sqrt{101}-10)^{6}\left(\dfrac{5\sqrt{5}+\sqrt{101}}{4} \pm \sqrt{\dfrac{105+5\sqrt{505}}{8}}\right)^{12}$

$\sqrt{38}$ $\quad \dfrac{g^3+g\sqrt{2}}{1+g^2\sqrt{2}} = \sqrt{1+\sqrt{2}}$

$\sqrt{26}$ $\quad \dfrac{g^3+g\cdot\dfrac{\sqrt{13}+1}{2}}{1+g^2\cdot\dfrac{\sqrt{13}-1}{2}} = \sqrt{\dfrac{3+\sqrt{13}}{2}}$

$\sqrt{50}$ $\quad \dfrac{g^3-g^2}{1+g} = \dfrac{\sqrt{5}+1}{2}$.

$\cfrac{a}{x+\cfrac{a^4}{x+\cfrac{a^8}{x+\cfrac{a^{12}}{x+\ldots}}}} = 1 - \cfrac{ax}{1+\cfrac{a^2}{1-\cfrac{a^2x}{1+\cfrac{a^4}{1-\ldots}}}}$ nearly.

$\left(\frac{1}{64}\right)$. $\sqrt{23}$. $1-g^3 = g^2$ }
$\sqrt{31}$ $1-g^3 = g$ }

(64.) $\sqrt{11}$ $\frac{1}{2}-g^3 = g-g^2$

$\sqrt{19}$ $\frac{1}{2}-g^3 = g^2$

$\sqrt{27}$ $\frac{1}{2}-g^3 = g^2\sqrt[3]{3}$.

$\sqrt{43}$ $\frac{1}{2}-g^3 = g$ $\sqrt{67}$ $\frac{1}{2}-g^3 = g^2+g$

$\sqrt{3}$ $2-\sqrt{3}$

$\sqrt{5}$ $\left(\sqrt{\frac{3+\sqrt{5}}{4}}-\sqrt{\frac{\sqrt{5}-1}{4}}\right)^4$

$\sqrt{7}$ $8-3\sqrt{7}$

$\sqrt{9}$ $\left(\sqrt{\frac{3+\sqrt{3}}{4}}-\sqrt{\frac{\sqrt{3}-1}{4}}\right)^8$

$\sqrt{13}$ $\left(\sqrt{\frac{7+\sqrt{13}}{4}}-\sqrt{\frac{3+\sqrt{13}}{4}}\right)^4$

$\sqrt{15}$ $(2-\sqrt{3})^2(4-\sqrt{15})$

$\sqrt{17}$ $\left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}}-\sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^8$

$\sqrt{21}$

$\sqrt{25}$ $\left(\sqrt{\frac{5+\sqrt{5}}{4}}-\sqrt{\frac{\sqrt{5}+1}{4}}\right)^8$

$\sqrt{7}$ ———— $8 - 3\sqrt{7}$

$\sqrt{15}$ ———— $(2-\sqrt{3})^2 (4-\sqrt{15})$

$\sqrt{39}$ ————

$\sqrt{55}$ ———— $(10-3\sqrt{11})(3\sqrt{5}-2\sqrt{11})\left(\sqrt{\frac{5+\sqrt{5}}{2}} - \sqrt{\frac{3+\sqrt{5}}{2}}\right)$