# MANUSCRIPT BOOK 3

# OF

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1. $a_1 e^{-b_1 x} + a_2 e^{-b_2 x} + a_3 e^{-b_3 x} + \cdots$

$$- \int^{\infty} a_z e^{-b_z x}\, dz$$ is finite when $x$ is 0?

2. $\frac{1}{a_1^{n+1}} + \frac{1}{a_2^{n+1}} + \frac{1}{a_3^{n+1}} + \cdots - \int \frac{dz}{a_z^{n+1}}$

is finite when $n$ is 0?

3. If $a_1 e^{-x} + a_2 e^{-2x} + a_3 e^{-3x} + \cdots - \frac{c}{x^r}$

is finite when $x$ is 0,

then the average value of $a_n$ is $\frac{c\, n^{r-1}}{\lfloor\underline{r-1}}$.

4. If $\int_0^\infty \phi(z) e^{-az}\, dz = \frac{1}{a^{n+1}}$ then $\phi(z) = \frac{z^n}{\lfloor\underline{n}}$.

5. i. The coeff.t of $x^{100}$ in $\frac{x^5}{(1-x^2)(1-x^3)}$ = coeff.t of $x^{95}$

in $\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)} = I\left(\frac{95}{2}\right) - I\left(\frac{95}{3}\right) = 16.$

ii. $I\left(\frac{n+4}{6}\right) - I\left(\frac{n+3}{6}\right) + I\left(\frac{n+2}{6}\right) = I\left(\frac{n}{2}\right) - I\left(\frac{n}{3}\right)$

~~proof~~. $\frac{x^2}{(1-x^2)(1-x^3)} = \frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)}$

or $\frac{x^2 - x^3 + x^4}{(1-x)(1-x^6)}$

iii. $I\left(\sqrt{n+1} + \sqrt{n}\right) = I\left(\sqrt{4n+2}\right)$.

iv. The coeff.t of $x^n$ in $\frac{\psi(x^2)}{1-x} = I\left(\frac{1}{2} + \sqrt{n+\frac{1}{2}}\right)$

$= I\left(\frac{1}{2} + \sqrt{n+\frac{1}{4}}\right)$.

6. If $N = a^p \cdot b^q \cdot c^r \cdots$ where $a, b, c$ are primes, the no. of divisors of $n$ is

$(p+1)(q+1)(r+1)(s+1) \cdots$

7. If $N$ is formed of the prime no $a$ and its powers alone then

i) The no of divisors will never exceed $\frac{\log_a N}{\log a}$

ii) if formed of primes $a, b$ then its will not exceed $\left(\frac{\log ab\, N}{2}\right)^2 \Big/ \log a \log b$.

iii) if formed of $a, b, c$ ..... $n$ primes the no of divisors will never exceed

$$\frac{\left\{\frac{\log (Nabc \text{------})}{n}\right\}^n}{\log a \log b \log c \text{.......}}$$

8. If $A, B, C$ are quantities so taken that

$$\frac{1}{A^k} + \frac{1}{B^k} + \frac{1}{C^k} + \text{.......} = \frac{a}{(k-\alpha)^n}$$

when $k = \alpha$, (the only pole) then the no of such quantities less than $x$ is

$$\int \frac{a (\log z)^{n-1}}{z^{1-\alpha} \lfloor n-1} dz$$

For $\int^\infty \frac{dn}{z^k} = \frac{a}{(k-\alpha)^n}$ and $\int^\infty \frac{dz}{z^{k-\alpha+1}} = \frac{1}{k-\alpha}$

Differentiating $n$ times with respect to $k$. we get the above result.

9. No. of the form $p^2+q^2$.

$$\frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{8^k}+\cdots$$

$$=\frac{1}{1-2^{-k}}\cdot\frac{1}{1-5^{-k}}\cdot\frac{1}{1-13^{-k}}\cdot\frac{1}{1-17^{-k}}\cdots$$

$$\times\frac{1}{1-3^{-2k}}\cdot\frac{1}{1-7^{-2k}}\cdot\frac{1}{1-11^{-2k}}\cdots$$

$$=\frac{\sqrt{S_k\cdot S'_k}}{1-2^{-k}}\sqrt{\frac{1}{1-3^{-2k}}\cdot\frac{1}{1-7^{-2k}}\cdot\frac{1}{1-11^{-2k}}\cdots}$$

where $S_k=\frac{1}{1^k}+\frac{1}{3^k}+\frac{1}{5^k}+\cdots$

and $S'_k=\frac{1}{1^k}-\frac{1}{3^k}+\frac{1}{5^k}-\cdots$

$$\frac{S_k}{S'_k}=\frac{1+3^{-k}}{1-3^{-k}}\cdot\frac{1+7^{-k}}{1-7^{-k}}\cdot\frac{1+11^{-k}}{1-11^{-k}}\cdots$$

Hence the series =

$$\frac{S'_k}{1-2^{-k}}\sqrt{\frac{S_k}{S'_k}}\cdot\sqrt[4]{\frac{S_{2k}}{S'_{2k}}}\sqrt[8]{\frac{S_{4k}}{S'_{4k}}}\sqrt[16]{\frac{S_{8k}}{S'_{8k}}}\cdots$$

$$=\frac{A}{\sqrt{k-1}}+\frac{B}{\sqrt[4]{2k-1}}+\frac{C}{\sqrt[8]{4k-1}}+\frac{D}{\sqrt[16]{8k-1}}+\cdots$$

where $A=\sqrt{\dfrac{\pi}{2(1-\frac{1}{3^2})(1-\frac{1}{7^2})(1-\frac{1}{11^2})\cdots}}$ and

$B, C, D$ are depending upon $A$.

Hence the reqd. nos. between $m$ and $n$

is $C\int_n^m\frac{dx}{\sqrt{\log x}}+\theta(x)$ where $C=\dfrac{1}{\sqrt{2(1-\frac{1}{3^2})(1-\frac{1}{7^2})\cdots}}$

and $\theta(x)$ is of the order $\dfrac{\sqrt{x}}{(\log x)^{\frac{3}{2}}}$. . .

obs. $\sqrt{2(1-\frac{1}{3^2})(1-\frac{1}{7^2})(1-\frac{1}{11^2})(1-\frac{1}{19^2})}=(1+\frac{1}{5})(1+\frac{1}{13})(1+\frac{1}{17})$

$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots\dots = a_1 - \underset{n=\infty}{Lt}\, a_n$$

$$\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \frac{a_3}{a_4} \dots\dots\dots = \underset{n=\infty}{Lt}\left(\frac{a_1}{a_n}\right).$$

$$\frac{1}{x} - \cot x = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + \frac{1}{8}\tan\frac{x}{8} + \dots$$

$$\frac{1}{2x} - \frac{1}{4}\cot\frac{x}{2} = \frac{\sin\frac{x}{3}}{3\left(1 + 2\cos\frac{x}{3}\right)} + \frac{\sin\frac{x}{9}}{9\left(1 + 2\cos\frac{x}{9}\right)} + \dots$$

$$\frac{3}{4}\sin x = \sin^3 x + \frac{\sin^3 3x}{3} + \frac{\sin^3 9x}{9} + \dots$$

$$\frac{1}{x} - \cot x = \left(\frac{1}{\sin x} - \frac{1}{x}\right) + \left(\frac{1}{\sin\frac{x}{2}} - \frac{2}{x}\right) + \left(\frac{1}{\sin\frac{x}{4}} - \frac{4}{x}\right) + \dots\dots$$

$$\frac{2x}{\sin 2x} = \frac{\tan x}{x} \cdot \left(\frac{2}{x}\tan\frac{x}{2}\right)^2 \left(\frac{4}{x}\tan\frac{x}{4}\right)^4 \dots$$

$$\frac{1}{\log x} + \frac{1}{1-x} = \frac{1}{2(1+\sqrt{x})} + \frac{1}{4(1+\sqrt[4]{x})} + \frac{1}{8(1+\sqrt[8]{x})} + \dots$$

$$\frac{1}{\log x} + \frac{1}{1-x} = \frac{2 + \sqrt[3]{x}}{3\left(1+\sqrt[3]{x}+\sqrt[3]{x^2}\right)} + \frac{2 + \sqrt[9]{x}}{9\left(1+\sqrt[9]{x}+\sqrt[9]{x^2}\right)} + \dots$$

$$\left\{1^2\log\left(1+\frac{x^2}{1^2}\right) - x^2\right\} + \left\{2^2\log\left(1+\frac{x^2}{2^2}\right) - x^2\right\} + \left\{3^2\log\left(1+\frac{x^2}{3^2}\right) - x^2\right\} + \dots\dots = \frac{x^2}{2} - \frac{\pi x^3}{3}$$

$$+ \frac{x}{\pi}\left(\frac{e^{-2\pi x}}{1^2} + \frac{e^{-4\pi x}}{2^2} + \frac{e^{-6\pi x}}{3^2} + \dots\right)$$

$$- x^2\log\left(1 - e^{-2\pi x}\right) - \frac{1}{2\pi^2}\left\{\frac{1 - e^{-2\pi x}}{1^3} + \frac{1 - e^{-4\pi x}}{2^3} + \frac{1 - e^{-6\pi x}}{3^3} + \dots\dots\right\}$$

$$\frac{1}{2}\left\{\log\left(1+\frac{1}{n}\right)\right\}^2 = \frac{\frac{1}{n}-\frac{1}{n+1}}{1^2} - \frac{\frac{1}{n^2}+\frac{1}{(n+1)^2}}{2^2} + \frac{\frac{1}{n^3}-\frac{1}{(n+1)^3}}{3^2} - \frac{\frac{1}{n^4}+\frac{1}{(n+1)^4}}{4^2} + \cdots$$

If $x = \left(\frac{\log\frac{1+\sqrt{5}}{2}}{\pi}\right)^2$ then $e^{\frac{x}{x}} = \frac{1+x}{e^x}\cdot\frac{\left(1+\frac{x}{4}\right)^4}{e^x}\cdot\frac{\left(1+\frac{x}{9}\right)^9}{e^x}\cdots$

$$\underline{|x}\; e^{Cx} = \frac{e^x}{1+x}\cdot\frac{e^{\frac{x}{2}}}{1+\frac{x}{2}}\cdot\frac{e^{\frac{x}{3}}}{1+\frac{x}{3}}\cdot\frac{e^{\frac{x}{4}}}{1+\frac{x}{4}}\cdots$$

If $\phi(x) = \frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}+\cdots$ then $\int\frac{\log(p+qx)}{r+sx}\,dx$, $\int\log(p+qx)\log(r+sx)\,dx$ and similar integrals, as well as the values of $\phi(\frac{1}{2})-\frac{1}{6}\phi(\frac{1}{9})$, $\phi(-\frac{1}{2})+\frac{1}{6}\phi(\frac{1}{9})$, $\phi(\frac{1}{4})+\frac{1}{3}\phi(\frac{1}{9})$, $\phi(-\frac{1}{3})-\frac{1}{3}\phi(\frac{1}{9})$, $\phi(\frac{1}{3})+\phi(\frac{1}{9})$, &c can be found. $\int_0^1 \frac{\log\frac{1+\sqrt{1+4a}}{2}}{a}\,da = \frac{\pi^2}{15}$. &c

$$\sqrt{1+a}\left\{1+\frac{ae^{-x}}{1-e^{-x}}+\frac{a^2e^{-3x}}{(1-e^{-x})(1-e^{-2x})}+\cdots\right\}$$
$$= e^{\frac{1}{x}\left(\frac{a}{1^2}-\frac{a^2}{2^2}+\frac{a^3}{3^2}-\cdots\right)+\frac{B_2}{\underline{|2}}\cdot\frac{x}{1+a}\cdot a - \frac{B_4}{\underline{|4}}\cdot\left(\frac{x}{1+a}\right)^3(a-a^2)+\frac{B_6}{\underline{|6}}\left(\frac{x}{1+a}\right)^5(a-11a^2+11a^3-a^4) - \&c}\ ?$$

$$1+\frac{ae^{-x}}{1-e^{-x}}+\frac{a^2e^{-4x}}{(1-e^{-x})(1-e^{-2x})}+\frac{a^3e^{-9x}}{(1-e^{-x})(1-e^{-2x})(1-e^{-3x})}+\cdots$$
$$= \sqrt{\frac{1+b}{1+2b}}\; e^{\frac{1}{x}\left\{\frac{1}{2}\left(\log\overline{1+b}\right)^2+\frac{b}{1^2}-\frac{b^2}{2^2}+\frac{b^3}{3^2}-\&c\right\}}$$

where $b+b^2=a$

If $a z^{2n} + z = 1$, then

when $x$ is very small the value of the series

$$1 + \frac{a e^{-nx-mx}}{1-e^{-x}} + \frac{a^2 e^{-4nx-2mx}}{(1-e^{-x})(1-e^{-2x})}$$

$$+ \frac{a^3 e^{-9nx-3mx}}{(1-e^{-x})(1-e^{-2x})(1-e^{-3x})} + \cdots$$

$$= \frac{z^m e^{\frac{1}{x}\int_0^{\log\frac{1}{z}} \frac{\log z}{a} da}}{\sqrt{z+2n(1-z)}} + (Ax + Bx^2 + \cdots)$$

$$\frac{2\pi}{\log 2} = 9.0647203; \quad \log\left(\frac{2\pi}{\log 2}\right) = 2.20437894; \quad \frac{2\pi^2}{\log 2} = 28.4776587$$

$$\log 2 \left\{ e^{-x} + 2e^{-2x} + 4e^{-4x} + 8e^{-8x} + \right.$$
$$\left. + 1 - \frac{x}{3\,\lfloor 1} + \frac{x^2}{7\,\lfloor 2} - \frac{x^3}{15\,\lfloor 3} + \frac{x^4}{31\,\lfloor 4} - \&c \right\}$$

$$= \frac{1 + .0000098844 \cos\left(\frac{2\pi \log x}{\log 2} + .872811\right)}{x}$$

If $(a_1 - a_2 + a_3 - \&c)(b_1 - b_2 + b_3 - \&c) = u_1 - u_2 + \cdots$

then $u_n$ is known when $n$ is very great

from (1) $\int_c^\infty u_n e^{-nh} dn = \int_c^\infty a_n e^{-nh} dn \times$

$\int_0^\infty b_n e^{-nh} dn$ where $h$ is a very small quantity

(2) $\int_0^{n-1} a_{1+z} b_{n-z} dz = u_n \quad \Big| \quad \frac{a_n \int b_n dn}{u_n}$ or

$\frac{\int a_n dn \int b_n dn}{n u_n}$ is finite when $n = \infty$ ??

The equation $\left.\begin{array}{l} x^2 = a + y \\ y^2 = a + z \\ z^2 = a + u \\ u^2 = a + x \end{array}\right\}$

is a 16th degree equation which can be reduced to four quartics of which one quartic is known by inspection and each of the three remaining quartics is of the form $\left(x^2 + px + \frac{p^2 - 2a - \frac{4}{p}}{2}\right)\left(x^2 + qx + \frac{q^2 - 2a - \frac{4}{q}}{2}\right)$ where $pq = -1$ and $p + q$ is a root of the equation $Z^3 + 3Z = 4(1 + aZ)$.

or $p$ is a root of $Z^6 - 4aZ^4 - 4Z^3 + 4aZ^2 - 1 = 0$

$$\sqrt{5 + \sqrt{5 + \sqrt{5 - \sqrt{5 + \sqrt{5 + \sqrt{5 + \sqrt{5 - \&c}}}}}}}$$
$$= \frac{2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}}}{2}$$

$$\sqrt{5 + \sqrt{5 - \sqrt{5 - \sqrt{5 + \sqrt{5 + \sqrt{5 - \sqrt{5 - \sqrt{5 + \&c}}}}}}}}$$
$$= \frac{\sqrt{5} - 2 + \sqrt{13 - 4\sqrt{5}} + \sqrt{50 + 12\sqrt{5} - 2\sqrt{65 - 20\sqrt{5}}}}{4}$$

Similarly for the product of three or more series; the value of $u_n$ is found from the value of the series $u_1 e^{-x} + u_2 e^{-2x} + \ldots$ when $x = 0$; e.g. $\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left(1 - \frac{1}{2} + \frac{1}{3} - \&c\right)$ is found from $u_1 e^{-x} + u_2 e^{-2x} + u_3 e^{-3x} + \ldots$
$$= \left(\frac{e^{-x}}{\sqrt{1}} + \frac{e^{-2x}}{\sqrt{2}} + \frac{e^{-3x}}{\sqrt{3}} + \ldots\right)\left(\frac{e^{-x}}{1} + \frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} + \ldots\right).$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 12 | 14 | 15 | 16 | 18 | 20 | 21 | 24 | 25 | 27 |
| 28 | 30 | 32 | 35 | 36 | 40 | 42 | 45 | 48 | 49 |
| 50 | 54 | 56 | 60 | 63 | 64 | 70 | 72 | 75 | 80 |
| 81 | 84 | 90 | 96 | 98 | 100 | 105 | 108 | 112 | 120 |
| 125 | 126 | 128 | 135 | 140 | 144 | 147 | 150 | 160 | 162 |
| 168 | 175 | 180 | 189 | 192 | 196 | 200 | 210 | 216 | 224 |
| 225 | 240 | 243 | 245 | 250 | 252 | 256 | 270 | 280 | 288 |
| 294 | 300 | 315 | 320 | 324 | 336 | 343 | 350 | 360 | 375 |
| 378 | 384 | 392 | 400 | 405 | 420 | 432 | 441 | 448 | 450 |
| 480 | 486 | 490 | 500 | 504 | 512 | 525 | 540 | 560 | 567 |
| 576 | 588 | 600 | 625 | 630 | 640 | 648 | 672 | 675 | 686 |
| 700 | 720 | 729 | 735 | 750 | 756 | 768 | 784 | 800 | 810 |
| 840 | 864 | 875 | 882 | 896 | 900 | 945 | 960 | 972 | 980 |
| 1000 | 1008 | 1024 | 1029 | 1050 | 1080 | 1120 | 1125 | 1134 | 1152 |
| 1176 | 1200 | 1215 | 1225 | 1250 | 1260 | 1280 | 1296 | 1323 | 1344 |
| 1350 | 1372 | 1400 | 1440 | 1458 | 1470 | 1500 | 1512 | 1536 | 1568 |
| 1575 | 1600 | 1620 | 1680 | 1701 | 1715 | 1728 | 1750 | 1764 | 1792 |
| 1800 | 1875 | 1890 | 1920 | 1944 | 1960 | 2000 | 2016 | 2025 | 2048 |
| 2058 | 2100 | 2160 | 2187 | 2205 | 2240 | 2250 | 2268 | 2304 | 2352 |
| 2400 | 2401 | 2430 | 2450 | 2500 | 2520 | 2560 | 2592 | 2625 | 2646 |
| 2688 | 2700 | 2744 | 2800 | 2835 | 2880 | 2916 | 2940 | 3000 | 3024 |
| 3072 | 3087 | 3125 | 3136 | 3150 | 3200 | 3240 | 3360 | 3375 | 3402 |
| 3430 | 3456 | 3500 | 3528 | 3584 | 3600 | 3645 | 3675 | 3750 | 3780 |
| 3840 | 3888 | 3920 | 3969 | 4000 | 4032 | 4050 | 4096 | 4116 | 4200 |
| 4320 | 4374 | 4375 | 4410 | 4480 | 4500 | 4536 | 4608 | 4704 | 4725 |
| 4800 | 4802 | 4860 | 4900 | 5000 | 5040 | 5103 | 5120 | 5145 | 5184 |
| 5250 | 5292 | 5376 | 5400 | 5488 | 5600 | 5625 | 5670 | 5760 | 5832 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5880 | 6000 | 6048 | 6075 | 6125 | 6144 | 6174 | 6250 | 6272 | 6300 |
| 6400 | 6480 | 6561 | 6615 | 6720 | 6750 | 6804 | 6860 | 6912 | 7000 |
| 7056 | 7168 | 7200 | 7203 | 7290 | 7350 | 7500 | 7560 | 7680 | 7776 |
| 7840 | 7875 | 7938 | 8000 | 8064 | 8100 | 8192 | 8232 | 8400 | 8505 |
| 8575 | 8640 | 8748 | 8750 | 8820 | 8960 | 9000 | 9072 | 9216 | 9261 |
| 9375 | 9408 | 9450 | 9600 | 9604 | 9720 | 9800 | 10000 | 10080 | 10125 |
| 10206 | 10240 | 10290 | 10368 | 10500 | 10584 | 10752 | 10800 | 10935 | 10976 |
| 11025 | 11200 | 11250 | 11340 | 11520 | 11664 | 11760 | 11907 | 12000 | 12005 |

Let $\sqrt[7]{1} = 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ such that

$\alpha + \alpha^2 + \alpha^4$ & $\alpha^3 + \alpha^6 + \alpha^5$ are $\frac{-1 \pm i\sqrt{7}}{2}$

$$\phi(x) = \sin x + \sin x\alpha + \sin x\alpha^2 + \&c$$
$$= -7\left(\frac{x^7}{\lfloor 7} - \frac{x^{21}}{\lfloor 21} + \frac{x^{35}}{\lfloor 35} - \&c\right)$$

$64 \sin x \sin x\alpha \sin x\alpha^2$ &c

$$= -\phi(2x) - \phi(2x.\overline{\alpha+\alpha^2+\alpha^4}) - \phi(2x.\overline{\alpha^3+\alpha^6+\alpha^5})$$
$$+ \phi(2x.\overline{\alpha+\alpha^6}) + \phi(2x.\overline{\alpha^2+\alpha^5}) + \phi(2x.\overline{\alpha^3+\alpha^4})$$
$$+ \phi\left(\frac{2x}{\alpha+\alpha^6}\right) + \phi\left(\frac{2x}{\alpha^2+\alpha^5}\right) + \phi\left(\frac{2x}{\alpha^3+\alpha^4}\right)$$

Similarly $\psi(x) = \cos x +$

$\psi(x) + \psi(\overline{2x.\alpha+\alpha^2+\alpha^4}) + \&c$ all plus.

$= 64 \cos x . \cos x\alpha \cos x\alpha^2$ &c - - -

from which 7 integral formula can be found

6 integral formula

If $a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$

so that $a_m a_n = a_{m+n} + a_{m-n}$, then

$$6\left(B_0 + B_{12}\frac{x^{12}}{\lfloor 13} + B_{24}\frac{x^{24}}{\lfloor 24} + \&c\right)$$

$$= \frac{\dfrac{x^5}{\lfloor 5}(a_3 + 2^3) - \dfrac{x^{17}}{\lfloor 17}(a_9 + 2^9) + \&c}{\dfrac{x^6}{\lfloor 6}(a_3 + 2^3) - \dfrac{x^{18}}{\lfloor 18}(a_9 + 2^9) + \&c}$$

$$6\left(B_2\frac{x^2}{\lfloor 2} + B_{14}\frac{x^{14}}{\lfloor 14} + B_{26}\frac{x^{26}}{\lfloor 26} + \&c\right)$$

$$= x \cdot \frac{\dfrac{x^7}{\lfloor 7}(a_4 + 2^4) - \dfrac{x^{19}}{\lfloor 19}(a_{10} + 2^{10}) + \ldots}{\dfrac{x^6}{\lfloor 6}(a_3 + 2^3) - \dfrac{x^{18}}{\lfloor 18}(a_9 + 2^9) + \&c}$$

$$6\left(B_6\frac{x^6}{\lfloor 6} + B_{18}\frac{x^{18}}{\lfloor 18} + \&c\right)$$

$$= x \cdot \frac{\dfrac{x^{11}}{\lfloor 11}(a_5 - 2^6) - \dfrac{x^{23}}{\lfloor 23}(a_{11} - 2^{12}) + \&c}{\dfrac{x^6}{\lfloor 6}(a_3 + 2^3) - \dfrac{x^{18}}{\lfloor 18}(a_9 + 2^9) + \&c}$$

$$6\left(B_8\frac{x^8}{\lfloor 8} + B_{20}\frac{x^{20}}{\lfloor 20} + \&c\right)$$

$$= x \cdot \frac{\dfrac{x^{13}}{\lfloor 13}(a_6 - 2^7) - \dfrac{x^{25}}{\lfloor 25}(a_{12} - 2^{13}) + \&c}{\dfrac{x^6}{\lfloor 6}(a_3 + 2^3) - \dfrac{x^{18}}{\lfloor 18}(a_9 + 2^9) + \&c}$$

$$\frac{x}{1+x} - \frac{x^3}{1+x^3} + \frac{x^5}{1+x^5} - \frac{x^7}{1+x^7} + \&c$$

$$= \frac{1}{4} + \frac{1}{4\sqrt{2}}\log(1+\sqrt{2}) - \frac{\pi}{8\sqrt{2}} \quad \text{when } x = 1$$

$$\frac{1^3 x}{1-x} - \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} - \frac{4^3 x^4}{1-x^4} + \ldots = \frac{1}{16}$$

(Examples). when $x = 1$.

# Numbers of the form $x^3 \pm y^3$.

1 2 7 8 9 16 19 26 27 28 35 37 54 56 61
63 64 65 72 91 98 117 124 125 126 127 128 133 152 169
189 208 215 216 217 218 224 243 250 271 279 280 296 316 331
335 341 342 343 344 351 370 386 387 397 407 432 448 469 485
488 504 511 512 513 520 539 547 559 576 602 604 631 637 657
665 686 702 721 728 729 730 737 756 784 793 817 819 854 855
866 875 919 936 945 973 989 992 999 1000.

## Long intervals of Composite nos

23 to 29; 89 to 97; 113 to 127; 523 to 541; 887 to 907;
1129 to 1151; 1327 to 1361; 9551 to 9587.
15683 to 15727; 19609 to 19661.
31397 to 31469; 265621 to 265703
360653 to 360749; 492113 to 492227
1561919 to 1562051. (below 2000000).
(370261 to 370373; 1357201 to 1357333; 2010733 to 2010881)

2 to 3; 3 to 5; and 7 to 11

## No. of primes

| | | |
|---|---|---|
| $10^5$ — 9592 | $6.10^5$ — 49098 | $2.10^6$ — 148931 |
| $2.10^5$ — 17984 | $7.10^5$ — 56543 | $3.10^6$ — 216816 |
| $3.10^5$ — 25997 | $8.10^5$ — 63951 | $10^7$ — 664579 |
| $4.10^5$ — 33860 | $9.10^5$ — 71274 | $10^8$ — 5761460 |
| $5.10^5$ — 41538 | $10^6$ — 78498 | $2.10^4$ — 2262 |

If P be any prime no. and there are k primes between P and $P + \phi(P, k)$, to find the max., min & average value of $\phi$.

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The highest composite no. can be found from

$$2^{\left[\frac{\log n}{\log 2}\right]-1}\, 3^{\left[\frac{\log n}{\log 3}\right]-1}\, 5^{\left[\frac{\log n}{\log 5}\right]-1}\, 7^{\left[\frac{\log n}{\log 7}\right]-1}$$

where $n$ is any positive quantity and [ ] mean-ing that the ~~integral~~ nearest integer is taken

The highest composite no. near the region $N$

is $2^{\frac{\log K}{\log 2}-1} \cdot 3^{\frac{\log K}{\log 3}-1} \cdot 5^{\frac{\log K}{\log 5}-1}\, 7^{\frac{\log K}{\log 7}-1}$

where $K = \dfrac{\log(N \cdot 2 \cdot 3 \cdot 5 \cdot \quad \theta K)}{n}$ where ?

$\theta K$ is a prime no. between $K$ and $\frac{K}{e}$

and $n$ the no. of primes from 2 to $\theta K$. ?

~~$K = \dfrac{\log(N \cdot 2 \cdot 3 \cdot 5 \ldots Z)}{n}$ where~~ ~~$Z$ is a prime just greater than $\log N$ and $n$ the no. of primes from 2 to $Z$~~ ?

The order of the no. of divisors of a highly composite no. $N = e^{\frac{\log N}{\log\log N}}$ ?

If $N$ be of the form $\lfloor a$ then the order is $e^{\frac{\log N}{(\log\log N)^2}}$

$$\frac{\chi(-x^2)\, f(-x^5)}{F(x, -x^5)} =$$

$$x \cdot \frac{\chi(-x^4)\, f(-x^5)}{f(-x^5, -x^5)}$$

$$\frac{f(-x,-x^5)}{f(-x^3,-x^3)} = \frac{1}{1+}\,\frac{x+x^2}{1+}\,\frac{x^2+x^4}{1+}\cdots$$

$$\begin{cases} \text{Num}^r = \dfrac{\phi(-x^3)}{f(-x)} \\[2ex] \text{Den}^r = \dfrac{\psi(x^3)}{f(-x^4)} \end{cases}$$

$$\frac{f(-x,-x^7)}{f(-x^3,-x^5)} = \frac{1}{1+}\,\frac{x+x^2}{1+}\,\frac{x^4}{1+}\,\frac{x^3+x^6}{1+}\cdots$$

If $\phi(a) = 1 + \dfrac{ax}{(1-x)(1-bx)} + \dfrac{a^2x^4}{(1-x)(1-x^2)(1-bx)(1-bx^2)}\cdots$

then $\dfrac{\phi(a)}{\phi(ax)} = 1 + \dfrac{ax}{1+}\,\dfrac{ax^2-bx}{1+}\,\dfrac{ax^3}{1+}\,\dfrac{ax^4-bx^2}{1+}\cdots$

If $u^n - u^{n-1} = v$

and $\displaystyle\int_0^1 \frac{\log u}{v}\,dv = \phi(n)$

then $\phi(0) = \dfrac{\pi^2}{6}$; $\phi(1) = \dfrac{\pi^2}{12}$; $\phi(2) = \dfrac{\pi^2}{15}$.

$$\phi(n) + \phi\left(\frac{1}{n}\right) = \frac{\pi^2}{6}.$$

$$\frac{f(x,x^7)}{f(-x^4,-x^{16})} = 1 + \frac{x}{1-x^4} + \frac{x^4}{(1-x^4)(1-x^8)} + \frac{x^9}{(1-x^4)\cdots}\cdots$$

$$x\frac{f(x^3,x^5)}{f(-x^8,-x^{12})} = x + \frac{x^4}{1-x^4} + \frac{x^9}{(1-x^4)(1-x^8)} + \frac{x^{16}}{(1-x^4)\cdots}\cdots$$

and the ratio

$$= \frac{x}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\,\frac{x^4}{1+}\cdots$$

If $\phi(a) = 1 + \dfrac{ax}{(1-x)(1+bx)} + \dfrac{a^2x^3}{(1-x)(1-x^2)(1+bx)(1+bx^2)}$

then $\dfrac{\phi(a)}{\phi(ax)} = 1 + \cfrac{ax}{1+\cfrac{bx}{1+\cfrac{ax^2}{1+\cfrac{bx^2}{1+\cfrac{ax^3}{1+\cfrac{bx^3}{1+}}}}}}$

If $x > 1$ $\quad \cfrac{1}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+}}}}$

oscillates between

$$1 - \cfrac{x^{-1}}{1+\cfrac{x^{-2}}{1-\cfrac{x^{-3}}{1+ \cdots}}}$$

and $\cfrac{x^{-1}}{1+\cfrac{x^{-4}}{1+\cfrac{x^{-8}}{1+\cfrac{x^{-12}}{1+}}}}$

$$e^{\frac{\pi}{4}\sqrt{30}} = 4\sqrt{3}\,(5+4\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{34}} = 12\,(4+\sqrt{17})$$

$$e^{\frac{\pi}{2}\sqrt{46}} = 12^2\,(147+104\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{42}} = 4\,(21+8\sqrt{6})$$

$$e^{\frac{\pi}{4}\sqrt{70}} = 12\sqrt{7}\,(5\sqrt{5}+8\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{78}} = 4\sqrt{3}\,(75+52\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{102}} = 4\sqrt{3}\,(200+49\sqrt{17})$$

$$e^{\frac{\pi}{4}\sqrt{130}} = 12\,(323+40\sqrt{65})$$

$$\pi = \frac{12}{\sqrt{130}} \log \left\{\frac{(3+\sqrt{13})(\sqrt{8}+\sqrt{10})}{2}\right\} \text{ to 15 dec.}$$

$$= \frac{24}{\sqrt{142}} \log \left(\frac{\sqrt{10+11\sqrt{2}} + \sqrt{10+7\sqrt{2}}}{2}\right) \text{ to 16 dec.}$$

$$= \frac{12}{\sqrt{190}} \log (3+\sqrt{10})(\sqrt{8}+\sqrt{10}) \text{ to 18 dec.}$$

$$\sqrt[4]{3^4 + 2^4 + \frac{1}{2+\left(\frac{2}{3}\right)^2}} = 3.14159265262\cdots$$

$$\pi = 3.14159265358\cdots$$

$$x + \frac{12}{\sqrt{n}} \left\{\frac{1}{\sinh \pi\sqrt{n}} + \frac{1}{2\sinh 2\pi\sqrt{n}} + \frac{1}{3\sinh 3\pi\sqrt{n}} + \right\}$$

$$\left.\begin{array}{l} \pi = 3.141592\cdots \\ \frac{9}{5} + \sqrt{\frac{9}{5}} = 3.14164\cdots \end{array}\right\} \text{error } .00005.$$

$$e^{\pi\sqrt{22}} = 2508951.9982$$

$$e^{\pi\sqrt{37}} = 199148647.999978$$

$$e^{\pi\sqrt{58}} = 24591257751.9999998222\cdots$$

$$\frac{63}{25}\cdot\frac{17+15\sqrt{5}}{7+15\sqrt{5}} = 3.14159265380$$

$$\frac{7}{3}\left(1+\frac{\sqrt{3}}{5}\right) = 3.14162$$

$$\frac{19}{16}\sqrt{7} = 3.14180$$

$$\sum p_n x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$$

$$n p_n = \sigma(1)\, p(n-1) + \sigma(2)\, p(n-2) + \sigma(3)\, p(n-3) + \cdots$$

---

$$\frac{1}{(1-a_1)(1-a_2)(1-a_3)\cdots}$$

$$= 1 + \frac{a_1 + a_2 + \cdots}{1-a_1} + \frac{a_2^2 + a_2 a_3 + a_3^2 + \cdots}{(1-a_1)(1-a_2)}$$

$$+ \frac{a_3^3 + a_3^2 a_4 \cdots}{(1-a_1)(1-a_2)(1-a_3)} + \frac{a_4^4 \cdots}{(1-a_1)(1-a_2)(1-a_3)}$$

$$\cancel{\frac{(x+a)^4-(x+b)^4}{2(a-b)}} = \cancel{(x+p)^3+(x+q)^3},$$

~~where~~ $\cancel{p = \frac{a}{3-\sqrt{3}} + \frac{b}{3+\sqrt{3}}}$

$\cancel{q = \frac{a}{3+\sqrt{3}} + \frac{b}{3-\sqrt{3}}}$

$$\left(x+\frac{1}{4}\right)^4 - \left(x-\frac{1}{4}\right)^4 = \left(x+\frac{1}{\sqrt{3}}\right)^3 + \left(x-\frac{1}{\sqrt{3}}\right)^3$$

If $\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$, then

$$(\alpha + \lambda^2\gamma)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + (\beta + \lambda^2\gamma)^3.$$

$$64\left\{(a+b+c)^6+(b+c+d)^6-(c+d+a)^6-(d+a+b)^6+(a-d)^6-(b-c)^6\right\}$$

$$\times\left\{(a+b+c)^{10}+(b+c+d)^{10}-(c+d+a)^{10}-(d+a+b)^{10}+(a-d)^{10}-(b-c)^{10}\right\}$$

$$=45\left\{(a+b+c)^8+(b+c+d)^8-(c+d+a)^8-(d+a+b)^8+(a-d)^8-(b-c)^8\right\}^2$$

---

If $3k = a^2+ab+b^2$, then

$$(ax^3+k)^3-(bx^3+k)^3 = k\left\{(x^2+ax)^3-(x^2+bx)^3\right\}$$

$$(x^4+1)^4+4\cdot\left(\frac{x^5-5x}{2}\right)^4+5(x^4-2)^4$$

$$=3^4+4\left(\frac{x^5+x}{2}\right)^4.$$

$$(4x^5-5x)^4+(2x^4+1)^4+5(4x^4-2)^4$$

$$=3^4+(4x^5+x)^4.$$

$$3^4+(2x^4-1)^4+(4x^5+x)^4$$

$$=(4x^4+1)^4+(6x^4-3)^4+(4x^5-5x)^4.$$

If $a+b+c=0$, then

(i) $2(ab+ac+bc)^2 = a^4+b^4+c^4$.

(ii) $2(ab+ac+bc)^4 = a^4(b-c)^4 + b^4(c-a)^4 + c^4(a-b)^4$.

(iii) $2(ab+ac+bc)^6 = (a^2b+b^2c+c^2a)^4 + (ab^2+bc^2+ca^2)^4 + (3abc)^4$.

(iv) $2(ab+ac+bc)^8 = (a^3+2abc)^4(b-c)^4 + (b^3+2abc)^4(c-a)^4 + (c^3+2abc)^4(a-b)^4$.

and so on.

If $\dfrac{a}{b} = \dfrac{c}{d}$, then

$$\left.\begin{aligned}&(a+b+c)^4 + (b+c+d)^4 + (a-d)^4\\ &= (c+d+a)^4 + (d+a+b)^4 + (b-c)^4\end{aligned}\right\}$$ 4 may be replaced by 2 also.

$2^4+4^4+7^4 = 3^4+6^4+6^4$.

$3^4+7^4+8^4 = 1^4+2^4+9^4$.

$6^4+9^4+12^4 = 2^4+2^4+13^4$.

$3^4+9^4 = 5^4+5^4+6^4+8^4$.

$2^4+2^4+7^4 = 4^4+4^4+5^4+6^4$

$3^4+9^4+14^4 = 7^4+8^4+10^4+13^4$

$7^4+10^4+13^4 = 5^4+5^4+6^4+14^4$

$1^4+2^4+4 \cdot 2^4 = 3^4$

$3^4+6^4+4 \cdot 4^4 = 7^4$

$7^4+8^4+4 \cdot 2^4 = 9^4$.

$3^4+14^4+4 \cdot 2^4 = 10^4+13^4$

$3^4+7^4+4 \cdot 2^4 = 5^4+5^4+6^4$.

$$4^4 + 6^4 + 8^4 + 9^4 + 14^4 = 15^4$$
$$1^4 + 2^4 + 12^4 + 24^4 + 44^4 = 45^4$$
$$4^4 + 21^4 + 22^4 + 26^4 + 28^4 = 35^4$$
$$4^4 + 8^4 + 13^4 + 28^4 + 54^4 = 55^4$$
$$1^4 + 8^4 + 12^4 + 32^4 + 64^4 = 65^4$$
$$22^4 + 28^4 + 63^4 + 72^4 + 94^4 = 105^4,$$
$$4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5$$
$$5^5 + 10^5 + 11^5 + 16^5 + 19^5 + 29^5 = 30^5$$
$$2^4 + 39^4 + 44^4 + 46^4 + 52^4 = 65^4$$
$$22^4 + 52^4 + 57^4 + 74^4 + 76^4 = 95^4$$

$$(8s^2 + 40st - 24t^2)^4 + (6s^2 - 44st - 18t^2)^4 + (14s^2 - 4st - 42t^2)^4 + (9s^2 + 27t^2)^4 + (4s^2 + 12t^2)^4 = (15s^2 + 45t^2)^4.$$

$$(4m^2 - 12n^2)^4 + (3m^2 + 9n^2)^4 + (2m^2 - 12mn - 6n^2)^4 + (4m^2 + 12n^2)^4 + (2m^2 + 12mn - 6n^2)^4 = (5m^2 + 15n^2)^4.$$

$$\sqrt[3]{2} = 1.259921,049894,873164,767208 = \frac{5}{4}\left(1 + \frac{24}{1000}\right)^{\frac{1}{3}} = \frac{63}{50}\left(1 + \frac{188}{1000000}\right)^{-\frac{1}{3}}$$

Contour integration

$$\int_0^\infty \frac{\cos nx}{1+x^2}\log x\,dx + \frac{\pi}{2}\int_0^\infty \frac{\sin nx}{1+x^2}dx = 0.$$

$(\log x)^2 \ldots\ldots\ldots \log x \ldots$

Elliptic function formulae.

$$\int_0^\infty e^{-x}x^{n-1}dx = \Gamma(n), \quad \int_{-\infty}^\infty \frac{n^{x-1}}{\Gamma(x)}dx = e^{n}.$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(x)}{n^x}dx = e^{-n}, \quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^x}{x^n}dx = \frac{1}{\Gamma(n)}$$

$$\int_0^\infty \cancel{x^{n-1}}\,\phi(x)\,dx = \psi(n), \quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} n^{-x}\psi(x)\,dx = \phi(n)$$

$$\int_{-\infty}^\infty n^{x-1}\phi(x)\,dx = \psi(n), \quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-n}\psi(x)\,dx = \phi(n).$$

$$\int_0^\infty \phi(x)e^{-nx}dx = \psi(n), \quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi(x)e^{nx}dx = \phi(n)$$

$$\int_{-\infty}^\infty \phi(x)e^{nx}dx = \psi(n), \quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi(x)e^{-nx}dx = \phi(n)$$

$$\int_0^\infty x^{n-1}\{\qquad\qquad\}dx = \Gamma(n)\phi(-n)$$

$$\int_{-\infty}^\infty \frac{\phi(x)}{\Gamma(x+1)}dx = \phi(0) + \frac{\phi(1)}{1!} + \frac{\phi(2)}{2!} + $$

$a_1^{a_2^{a_3}}$ is ~~convergent~~ convergent when

$$1 + \log\log a_n \le \frac{1}{2}\left\{\frac{1}{n^2} + \frac{1}{(n\log n)^2} + \frac{1}{(n\log n\log\log n)^2} + \cdots + \frac{1}{(n\log n\log\log n\log\log\log n)^2} \cdots\right\}$$

divergent when it is greater than the right-hand side when any 1 is replaced by $1+\varepsilon$.

$\frac{1}{4} + 2 = (1\frac{1}{2})^2$

$\frac{1}{4} + 2.3 = (2\frac{1}{2})^2$

$\frac{1}{4} + 2.3.5 = (5\frac{1}{2})^2$

$\frac{1}{4} + 2.3.5.7 = (14\frac{1}{2})^2$

$\frac{1}{4} + 2.3.5.7.11.13.17 = (714\frac{1}{2})^2$.

$1 + 2 = 3$

$2 + 3 = 5, \quad 1 + 5 = 2.3.$

$3 + 7 = 2.5, \quad 1 + 2.7 = 3.5.$

$2.5 + 11 = 3.7, \quad 3.\cancel{5}+7 = 2.11, \quad 2 + 3.11 = 5.7$

$2.3.7 + 13 = 5.11$

$3.5.11 + 17 = 2.7.13, \quad 1 + 2.3.7.17 = 5.11.13$

| | $8J+3$ | $64J^2-24J+9$ |
|---|---|---|
| 11 — | 11 | $7^2$ |
| 19 — | 27 | $27\cdot 19$ |
| 43 — | $27\cdot 3^2$ | $27\cdot 7^2\cdot 43$ |
| 67 — | $27\cdot 7^2$ | $27\cdot 31^2\cdot 67$ |
| 163 — | $27\cdot 77^2$ | $27\cdot 2413^2\cdot 163.$ |

$$\sqrt[6]{2\sqrt{64J^2-24J+9}-(16J-3)} = t = \sqrt{3}\, x^{1/18} \frac{f(x^3)f(x)}{f^2(x)}$$

11 — $t-1=0$

35 — $t^2+t-1=0$

59 — $t^3+2t-1=0$

83 — $t^3+2t^2+2t-1=0$

107 — $t^3-2t^2+4t-1=0$

$$\frac{1}{3}\sqrt{1+\frac{8}{3}J} = t \qquad \frac{e^{\pi\sqrt{n}/6}+6e^{-\pi\sqrt{n}/6}}{6\sqrt{3}} \text{ appr}$$

19 — $t-1=0$

43 — $t-3=0$

67 — $t-7=0$

91 — $t=7+2\sqrt{13}$. $\quad t^2-14t-3=0$

115 — $t=13+6\sqrt{5}$ $\quad t^2-26t-11=0$

163 — $t-77=0$

$$R = x^{1/12}\left(\frac{f(x)}{f(x^3)}\right)^{1/3}$$

$$\frac{3\sqrt{3}}{R^6} = \sqrt{8J+3} + \sqrt{2\sqrt{64J^2-24J+9}-8J+6}$$

$$J = \frac{1 - 16\alpha(1-\alpha)}{8\sqrt[3]{4\alpha(1-\alpha)}} \qquad J = \frac{\sqrt[3]{4t}}{2^3}$$

$J_3 = 0,\ J_{11} = 1,\ J_{19} = 3,\ J_{27} = 5\sqrt[3]{3},\ J_{35} = \sqrt{5}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^4$

$J_{43} = 30,\ J_{51} = 3\cdot\left(\frac{5+\sqrt{17}}{2}\right)\sqrt[3]{(4+\sqrt{17})^2}$

$$t = \frac{(1-p^8)^3}{p^8}$$

$J_{59} = (p^9 - 7p^8 + 22p^7 - 34p^6 + 40p^5 - 28p^4$
$+ 22p^3 - 10p^2 + 11p - 1) = 0$

cubic in $t$.

$J_{67} = 165,\ J_{75} = 3\cdot\left(\frac{69+31\sqrt{5}}{2}\right)\sqrt[6]{5}$

$J_{83}.\quad \beta = \frac{1 - 2s - 2s^2 - 2s^3}{2s^2}$

$\beta^3 + 4\beta^2 + 2\beta - 5 = 0$

$$t = \frac{(1 - 2^{16}s^{24})^3}{2^{16}s^{24}}$$

$J_{91} = 3\left(\frac{227+63\sqrt{13}}{2}\right)\quad J_{99} = \cdots$

$J_{115} = 3\left(\frac{785+351\sqrt{5}}{2}\right),\ J_{163} = 20010$

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$1 + 5\cdot\frac{1}{2}\cdot\frac{1.3}{4^2}\,4x(1-x) + \cdots$

$= \frac{1}{1-2x}\left\{1 - \left[\frac{4}{y}\right] - 8\left(\frac{1}{e^y-1} + \frac{2}{e^{2y}-1} + \cdots\right)\right.$
$\left. - 16\left(\frac{1}{e^{2y}-1} + \frac{1}{e^{5y}-1} + \cdots\right)\right\}$

$1 + 4\cdot\frac{1}{2}\cdot\frac{1.2}{3^2}\,4x(1-x) + \cdots$

$= \frac{1}{\cdots}\left\{1 - \left[\frac{3}{y}\right] - 6\left(\frac{1}{e^y-1} + \frac{2}{e^{2y}-1} + \cdots\right)\right.$
$\left. - 18\left(\frac{1}{e^{2y}-1} + \frac{\cdots}{e^{6y}-1} + \cdots\right)\right\}$

---

$1 + \frac{1}{3^2}\,c + \cdots$

$\sqrt[3]{c(1-c)} = \frac{2}{3}\cdot\frac{2 - \sqrt[6]{99'}}{\sqrt[6]{99'}} - \frac{2}{3}\cdot\frac{2 + \sqrt[6]{99'}}{\sqrt[6]{99'}}.$

$\underline{x = 2}$

$$2\left\{1-24\left(\frac{q^4}{1-q^4}+\frac{2q^8}{1-q^8}\cdots\right)\right\}$$
$$-\left\{1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}\cdots\right)\right\}$$
$$=\frac{4KL}{\pi^2}(k'+l)$$

$\underline{n = 4}$
$$4\left\{1-24\left(\frac{q^8}{1-q^8}+\frac{2q^{16}}{1-q^{16}}+\cdots\right)\right\}$$
$$-\left\{1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\cdots\right)\right\}$$
$$=\frac{12KL}{\pi^2}\left(\sqrt{k'}+\sqrt{l}\right)^2$$

If $x > 1$, then $\frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\cdots$ oscillates between

$$1-\frac{x-1}{1+}\,\frac{x-2}{1-}\,\frac{x-3}{1+}\cdots$$

and
$$\frac{x-1}{1+}\,\frac{x-4}{1+}\,\frac{x-8}{1+}\,\frac{x-12}{1+}\cdots$$

41. $z = x+\frac{4}{x}$ $\quad 4x = 6/2kk'$

$$k^2 - z,\ \frac{5+\sqrt{41}}{2} + \frac{7+\sqrt{41}}{2}$$

$$I\left(\frac{x}{1^n}\right)+I\left(\frac{x}{2^n}\right)+I\left(\frac{x}{3^n}\right)+\cdots$$
$$= xS_n + x^{\frac{1}{n}}S_{\frac{1}{n}} + O(x^{\epsilon}).$$
$$= I\left(\sqrt[n]{\tfrac{x}{1}}\right)+I\left(\sqrt[n]{\tfrac{x}{2}}\right)+I\left(\sqrt[n]{\tfrac{x}{3}}\right)+\cdots$$

~~If $a_1, a_2, a_3$ &c are increasing +ve integral numbers, then~~ and ~~$a_n - a_{n-1}$~~ is not ~~always finite when $n \to \infty$~~, then ~~$\frac{a_n}{n} \to \infty$~~

---

If the no. of such nos as $a_1, a_2, a_3, a_4, \ldots$ within $n = O\phi(n)$.

then $\cos a_1 x + \cos a_2 x + \cdots \cos a_n x$

$$= O\left\{\frac{n}{\phi(n)}\right\}.$$

The sum of the ~~no of~~ divisors of ~~the~~

$$= e^{\gamma} N (\log \log N) \;?$$

$$\frac{2^{p+1}-1}{2-1} \cdot \frac{3^{q+1}-1}{3-1} \cdot \frac{5^{r+1}-1}{5-1} \cdots$$

$$\frac{355}{113}\left(1 - \frac{.0003}{3533}\right) = 3.14159265358979432.$$

If $u = \frac{\sqrt[5]{x}}{1+} \frac{x}{1+} \frac{x^2}{1+} \frac{x^3}{1+} \cdots$

then $u^2 + u - 1 = 0$ when $x^n = 1$, where $n$ is any positive integer except multiples of 5 in which case $u$ is not definite.

$$\frac{27}{4\pi} = 2 + 17\cdot\frac{1}{2}\cdot\frac{1.2}{3^2}\cdot\left(\frac{2}{27}\right) + \cdots$$

$$\frac{15\sqrt{3}}{2\pi} = 4 + 37\cdot\frac{1}{2}\cdot\frac{1.2}{3^2}\cdot\left(\frac{4}{125}\right) + \cdots$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1 + 12\cdot\frac{1}{2}\cdot\frac{1.5}{6^2}\cdot\left(\frac{4}{125}\right) + \cdots$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8 + 141\cdot\frac{1}{2}\cdot\frac{1.5}{6^2}\left(\frac{4}{85}\right)^3 + \cdots$$

$$\frac{4}{\pi} = \frac{3}{2} - \frac{23}{2^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{4}{\pi\sqrt{3}} = \frac{3}{4} - \frac{31}{3.4^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{4}{\pi} = \frac{23}{18} - \frac{283}{18^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{4}{\pi\sqrt{5}} = \frac{41}{72} - \frac{685}{5.72^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{2}{\pi\sqrt{3}} = \frac{1}{3} + \frac{9}{3^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1}{9} + \frac{11}{9^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{1}{3\pi\sqrt{3}} = \frac{3}{49} + \frac{43}{49^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{2}{\pi\sqrt{11}} = \frac{19}{99} + \frac{299}{99^3}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}\cdot\frac{1}{2}\cdot\frac{1.3}{4^2} + \cdots$$

$1\left(\frac{n}{a}\right) + \frac{1}{2}\left(\frac{n}{a^2}\right) + \cdots$

$\frac{n-1}{a-1}$ and $\frac{n}{a-1} - \frac{\log(n+1)}{\log a}$

$a$ and $n$ being integers.