Zeros of the Riemann Zeta Function

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Abstract

In this paper, we present a proof of the Riemann hypothesis. We show that zeros of the Riemann zeta function should be on the line with the real value 1/2, in the region where the real part of complex variable is between 0 and 1.

1 Introduction

The Riemann zeta function is a function of complex variable which is defined in Definition 2.1. The Riemann hypothesis is that the Riemann zeta function has only non-trivial zeros on the critical line (i.e., the line with real value 1/2 in the complex plane). Trivial zeros of the function are negative even numbers. In this paper, we prove the Riemann hypothesis in four steps by dividing the full complex plane into four regions. In Theorem 2.3, we review the well-known result regarding the region where the real value of complex variable is bigger than 1, by following [1]. In Theorem 2.4, we review the fact that there is no zero on the line with real value 1, by following [1]. In Theorem 2.5, using Theorem 2.2, 3, 4, we investigate the region where its real value of complex variable is non-positive. In Theorem 3.11, we give the proof that there are zeros only on the critical line in the region where the real value of complex variable is between 0 and 1. Therefore, we cover the entire region of the complex plane.

2 Riemann Hypothesis and well-known facts

Definition 2.1. *The Riemann zeta function is a function of complex variable proposed by Riemann [2]*

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} & \text{if } Re(s) > 1, \\ \frac{1}{1-\frac{1}{2^{s-1}}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} & \text{if } Re(s) > 0 \end{cases}$$

which is not defined at s=0, 1.

Theorem 2.2. *The Riemann zeta function satisfies the functional equation which is stated below.*

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{s\pi}{2}) \Gamma(1-s) \zeta(1-s) \quad (\forall s \in \mathfrak{C}).$$

Proof. It is written in Theorem 1.6 of [1].

Theorem 2.3. *The Riemann zeta function has no zero in the region where the real value of s is bigger than 1.*

Proof. By equation (1.2) of [1]

$$\zeta(s) = \prod_p \frac{1}{(1-p^{-s})},$$

where p ranges over the primes. Therefore, we have

$$\zeta(s) \prod_p (1-p^{-s}) = 1.$$

All of the factors of this infinite product can be found in the product

$$\prod_{n=2}^{\infty} (1-n^{-s})$$

which converges absolutely, since the zeta sum

$$\sum_{k=1}^{\infty} k^{-s}$$

converges absolutely. Hence

$$\prod_p (1 - p^{-s})$$

converges absolutely, and so by the fact that

$$\zeta(s) \prod_p (1 - p^{-s}) = 1,$$

we know that the Riemann zeta function cannot possibly be zero for any point in the region in question. Therefore, the theorem is proved.

Theorem 2.4. *The Riemann zeta function has no zero on the line with real value 1.*

Proof. It is written in Theorem 1.5 of [1].

Theorem 2.5. *The Riemann zeta function has only the trivial zeros in the region where the real value of s is non-positive. The trivial zeros mean negative even numbers.*

Proof. According to Theorem 2.3 and Theorem 2.4, the Riemann zeta function has no zero in the region where the real value of s is bigger than or equal to 1. Moreover, by Theorem 2.2 the Riemann zeta function satisfy the following equation, so-called *functional equation* for the Riemann zeta function.

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{s\pi}{2}) \Gamma(1-s) \zeta(1-s) \quad (\forall s \in \mathfrak{C}).$$

Therefore the Riemann zeta function becomes zero in the region where the real value of s is non-positive only when the *sin* term is zero, because the *gamma* function Γ has no zero in the complex plane and the Riemann zeta function $\zeta(1-s)$ has no zero since $Re(1-s) \geq 1$. Since the *sin* function has zeros at natural multiple of π, we can conclude that the only zeros are negative even numbers in the region in question since the Riemann zeta function is not defined at $s = 0$. Therefore, the theorem is proved.

3 Proof of the Riemann Hypothesis

Definition 3.1. *We define a function* $\Omega(a)$ *of natural number* $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, *where* p_i *is a prime number*

$$\Omega(a) = \sum_{i=1}^{k} a_i.$$

Theorem 3.2. *The function* Ω *defined in Definition 3.1 has the following property for two integer* a, b *which is coprime for each other.*

$$\Omega(ab) = \Omega(a) + \Omega(b)$$

Proof. Since a and b is coprime we can state a and b as follows

$$a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

$$b = p_{k+1}^{a_{k+1}} p_{k+2}^{a_{k+2}} \cdots p_{k+t}^{a_{k+t}}.$$

then the equation we want to prove turns to be true since

$$\Omega(ab) = \sum_{i=1}^{k+t} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{k+t} a_i = \Omega(a) + \Omega(b).$$

Definition 3.3. *We define a function* $\beta(n)$ *of natural number* n

$$\beta(n) = \sum_{\forall m | n} (-1)^{\frac{n}{m}+1} (-1)^{\Omega(m)}.$$

Theorem 3.4. *The function* β *defined in Definition 3.3 has the following values for odd prime number* p

$$\beta(p^n) = \begin{cases} 1 & \text{if n is even number,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By using Definition 3.3 the following equation holds:

$$\beta(p^n) = \sum_{i=0}^{n} (-1)^{\frac{p^n}{p^i}+1}(-1)^{\Omega(p^i)} = \sum_{i=0}^{n} (-1)^{p^{n-i}+1}(-1)^i = \sum_{i=0}^{n} (-1)^i.$$

Therfore we have the following result:

$$\beta(p^{2k}) = 1\, and\, \beta(p^{2k+1}) = 0.$$

Theorem 3.5. *The function β defined in Definition 3.3 has the following property for odd number m and odd prime number p*

$$\beta(p^a m) = \beta(p^a)\beta(m) \text{ if } p \nmid m.$$

Proof. By using Definition 3.3 and the fact $(-1)^{p^{a-i}+1} = 1$:

$$\beta(p^a m) = \sum_{\forall t | p^a m} (-1)^{\frac{p^a m}{t}+1}(-1)^{\Omega(t)} = \sum_{i=0}^{a} \sum_{\forall t|m} (-1)^{\frac{p^a m}{p^i t}+1}(-1)^{\Omega(p^i t)}$$

$$= \sum_{i=0}^{a} \sum_{\forall t|m} (-1)^{p^{a-i}\frac{m}{t}+1}(-1)^{\Omega(p^i)}(-1)^{\Omega(t)} = \sum_{i=0}^{a} (-1)^i \sum_{\forall t|m} (-1)^{p^{a-i}\frac{m}{t}+1}(-1)^{\Omega(t)}$$

$$= \sum_{i=0}^{a} (-1)^i \sum_{\forall t|m} (-1)^{\frac{m}{t}+1}(-1)^{\Omega(t)} = \sum_{i=0}^{a} (-1)^i \beta(m) = \beta(m) \sum_{i=0}^{a} (-1)^i$$

$$= \beta(m) \sum_{i=0}^{a} (-1)^{\frac{p^a}{p^i}+1}(-1)^i = \beta(m)\beta(p^a).$$

Theorem 3.6. *The function β defined in Definition 3.3 has the following values for odd number n*

$$\beta(n) = \begin{cases} 1 & \text{if } n = m^2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By using Theorem 3.5 it is obvious that non square number has the value of function beta zero because the following equation holds when $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$:

$$\beta(n) = \beta(p_1^{a_1})\beta(p_2^{a_2}) \cdots \beta(p_k^{a_k}),$$

and the beta value of odd power number is zero by Theorem 3.4.

Theorem 3.7. *The function β defined in Definition 3.3 has the following values*

$$\beta(n) = \begin{cases} 1 & \text{if } n = m^2, \text{ where } m \in \mathbb{N}, \\ -2 & \text{if } n = 2m^2, \text{ where } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since the even numbers can be represented as $2^a m$ with an odd number m, we have the following equation:

$$\beta(2^a m) = \sum_{\forall t | 2^a m} (-1)^{\frac{2^a m}{t}+1} (-1)^{\Omega(t)} = \sum_{i=0}^{a} \sum_{\forall t|m} (-1)^{\frac{2^a m}{2^i t}+1} (-1)^{\Omega(2^i t)}$$

$$= \sum_{i=0}^{a} \sum_{\forall t|m} (-1)^{2^{a-i}\frac{m}{t}+1} (-1)^{\Omega(2^i)} (-1)^{\Omega(t)} = \sum_{i=0}^{a} (-1)^i \sum_{\forall t|m} (-1)^{2^{a-i}\frac{m}{t}+1} (-1)^{\Omega(t)}$$

$$= -\sum_{i=0}^{a-1} (-1)^i \sum_{\forall t|m} (-1)^{\frac{m}{t}+1} (-1)^{\Omega(t)} + (-1)^a \sum_{\forall t|m} (-1)^{\frac{m}{t}+1} (-1)^{\Omega(t)}$$

$$= -\sum_{i=0}^{a-1} (-1)^i \beta(m) + \beta(m)(-1)^a.$$

When m is not a square number, $\beta(2^a m) = 0$ by using Theorem 3.6 and the fact derived above.

When m is a square number, we have the result

$$\beta(2^a m) = \begin{cases} 1 & \text{if } a \text{ is even}, \\ -2 & \text{if } a \text{ is odd}, \end{cases}$$

where we have used the relation

$$\beta(2^a m) = \sum_{i=0}^{a-1} \beta(m)(-1)^i(-1) + \beta(m)(-1)^a = \sum_{i=0}^{a-1} (-1)^i(-1) + (-1)^a.$$

Therefore, the theorem is proved.

Lemma 3.8. *If all p,q,r and s are nonzero real numbers and*

$$\frac{q}{p} \neq \frac{s}{r},$$

then we can say the following equations have only solution that A=B=0:

$$pA + qB = 0,$$
$$rA + sB = 0.$$

Definition 3.9. *We define a function mrzf(a,b,t,σ) of natural numbers a, b and real numbers t, σ as*

$$mrzf(a,b,t,\sigma) = \frac{(-1)^{b+1} sin(tln(ab))}{(ab)^\sigma}(-1)^{\Omega(a)}.$$

Let's call this function as middle-riemann-zero-function.

Theorem 3.10. *The following equation holds for middle-riemann-zero-function mrzf:*

$$\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} mrzf(m,l,t,\sigma) = \sum_{n=1}^{\infty}\sum_{\forall m|n} mrzf(m,\frac{n}{m},t,\sigma)$$

if $\sum_{l=1}^{\infty} mrzf(m,l,t,\sigma)$ *converges.*

Proof. *First let's construct four sets A, B,* C_n *and* D_m *as follows.*

$$A = \{(m,l)|m,l \in \mathbb{N}\},$$
$$B = \{(m,\frac{n}{m})|m,n \in \mathbb{N}\&m|n\},$$
$$C_n = \{(m,\frac{n}{m})|m|n\},$$
$$D_m = \{(m,l)|l \in \mathbb{N}\}.$$

First, let's show that set A and B is equivalent. To prove this statement we have to prove that A is a subset of B and B is a subset of A. A is a subset of B since for every element $a = (a_1, a_2)$ *in A it is an element of B because when we let* $m = a_1$ *and* $n = a_1a_2$, *m and n satisfies the property to be the element of B and therefore* $(a_1, \frac{a_1a_2}{a_1}) = a$ *is an element of B. Therefore A is a subset of B. Similarily, B is a subset of A since for every element* $b = (b_1, b_2)$ *in B it is an element of A since* $b_1, b_2 \in \mathbb{N}$ *because b satisfies the property to be an element of B. Therefore B is a subset of A and thus A is equivalent to B.*

Secondly, let's show that $\bigcup_{n=1}^{\infty} C_n = B$ *and* $\bigcup_{m=1}^{\infty} D_m = A$.

$$\forall b = (p, \frac{q}{p}) \in B, p, q \in \mathbb{N}\, and\, p \mid q\, and\, therefore$$

$$b \in C_q\, and\, thus\, b \in \bigcup_{n=1}^{\infty} C_n$$

$$\forall c = (p, \frac{q}{p}) \in \bigcup_{n=1}^{\infty} C_n, \exists q\, s.t.\, c \in C_q$$

$$and\, thus\, p, q \in \mathbb{N}\, and\, p \mid q\, and\, therefore\, c \in B.$$

$$\therefore B = \bigcup_{n=1}^{\infty} C_n.$$

$$\forall a = (p, q) \in A, p, q \in \mathbb{N}\, and\, therefore$$

$$a \in D_p\, and\, thus\, a \in \bigcup_{m=1}^{\infty} D_m$$

$$\forall d = (p, q) \in \bigcup_{m=1}^{\infty} D_m, \exists p\, s.t.\, d \in D_p$$

$$and\, thus\, p, q \in \mathbb{N}\, and\, therefore\, d \in A.$$

$$\therefore A = \bigcup_{m=1}^{\infty} D_m.$$

Finally, we can state the following equation by the definition of the sets.

$$\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} mrzf(m,l,t,\sigma)$$

$$= \sum_{m=1}^{\infty}\sum_{(m,l)\in D_m} mrzf(m,l,t,\sigma) = \sum_{(m,l)\in\bigcup_{m=1}^{\infty} D_m} mrzf(m,l,t,\sigma)$$

$$= \sum_{(m,l)\in A} mrzf(m,l,t,\sigma) = \sum_{(m,\frac{n}{m})\in B} mrzf(m,\frac{n}{m},t,\sigma)$$

$$= \sum_{(m,\frac{n}{m})\in\bigcup_{n=1}^{\infty} C_n} mrzf(m,\frac{n}{m},t,\sigma) = \sum_{n=1}^{\infty}\sum_{(m,\frac{n}{m})\in C_n} mrzf(m,\frac{n}{m},t,\sigma)$$

$$= \sum_{n=1}^{\infty}\sum_{\forall m|n} mrzf(m,\frac{n}{m},t,\sigma).$$

Therefore, the theorem is proved.

Since

$$\because \sum_{(m,l)\in D_m} mrzf(m,l,t,\sigma) = 0$$

$$\text{and} \sum_{(m,\frac{n}{m})\in C_n} mrzf(m,\frac{n}{m},t,\sigma) = \sum_{(m,\frac{n}{m})\in C_n} \frac{(-1)^{\frac{n}{m}+1} sin(tln(n))}{(n)^{\sigma}}(-1)^{\Omega(m)}$$

$$= \frac{\beta(n)sin(tln(n))}{n^{\sigma}} \begin{cases} \frac{sin(tln(n))}{n^{\sigma}} & \text{if } n = m^2, \text{ where } m\in\mathbb{N}, \\ -2\frac{sin(tln(n))}{n^{\sigma}} & \text{if } n = 2m^2, \text{ where } m\in\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

converges.

Theorem 3.11. *The Riemann zeta function only has zeros with real value 1/2 in the region where the real value of the complex variable s is between 0 and 1.*

Proof. *Let us assume that there is a zero s in the region where Re(s) is between 0 and 1 and that Re(s) is not equal to 1/2. Write the zero as* $s = \sigma + jt$ *with an imaginary number* j *whose square is* -1*. By Theorem 2.2, since the sin term and the gamma term* Γ *never equals to zero when Re(s) is between 0 and 1, 1-s is a zero too. Therefore, without loss of generality, we can assume that* σ *is bigger than 1/2.*

Since s is a zero of the Riemann zeta function and Re(s) is smaller than 1, σ and t satisfy the following equation by Definition 2.1:

$$\zeta(\sigma + jt) = \frac{1}{1 - \frac{1}{2^{\sigma+jt-1}}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\sigma+jt}} = 0.$$

Since $\frac{1}{1-\frac{1}{2^{\sigma+jt-1}}} \neq 0$*, we get the equation*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\sigma+jt}} = 0.$$

Using relations $n^{-jt} = e^{-t\ln(n)j} = \cos(-t\ln(n)) + j\sin(-t\ln(n))$ *('ln' means natural logarithm) we can rewrite the above equation as*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-t\ln(n)j}}{n^{\sigma}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\cos(t\ln(n)) - j\sin(t\ln(n)))}{n^{\sigma}} = 0,$$

which becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(t\ln(n))}{n^{\sigma}} = 0, \qquad (3.1)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(t\ln(n))}{n^{\sigma}} = 0. \qquad (3.2)$$

By multiplying equations equation (3.1) and equation (3.2) by sinϕ and cosϕ, respectively, and adding together, we get the equation

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(t\ln(n) + \phi)}{n^{\sigma}} = 0, \qquad (3.3)$$

where we have used the relation $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. *Letting* $\phi = t\ln(m)$ *and multiplying equation (3.3) by* $m^{-\sigma}$*, we get the identity*

$$(f_1(m) \equiv) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(t\ln(mn))}{(mn)^{\sigma}} = 0.$$

By above definition of $f_1(m)$ and Theorem 3.10, we get:

$$
\begin{aligned}
0 &= \sum_{m=1}^{\infty} 0(-1)^{\Omega(m)} \\
= \sum_{m=1}^{\infty} f_1(m)(-1)^{\Omega(m)} &= \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{(-1)^{l+1}\mathrm{sin}(t\mathrm{ln}(ml))}{(ml)^{\sigma}}(-1)^{\Omega(m)} \\
= \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} mrzf(m,l,t,\sigma) &= \sum_{n=1}^{\infty}\sum_{\forall m|n} mrzf(m,\frac{n}{m},t,\sigma) \\
&= \sum_{n=1}^{\infty}\sum_{\forall m|n} \frac{(-1)^{\frac{n}{m}+1}\mathrm{sin}(t\mathrm{ln}(m\frac{n}{m}))}{(m\frac{n}{m})^{\sigma}}(-1)^{\Omega(m)} \\
&= \sum_{n=1}^{\infty}\sum_{\forall m|n} \frac{(-1)^{\frac{n}{m}+1}sin(tln(n))}{n^{\sigma}}(-1)^{\Omega(m)} \\
&= \sum_{n=1}^{\infty} \frac{\beta(n)\mathrm{sin}(t\mathrm{ln}(n))}{n^{\sigma}} = 0
\end{aligned}
$$

since $f_1(m)(-1)^{\Omega(m)} = \sum_{l=1}^{\infty} mrzf(m,l,t,\sigma) = 0$ *converges.*

Which becomes

$$\sum_{n=1}^{\infty} \frac{\mathrm{sin}(t\mathrm{ln}(n^2))}{(n^2)^{\sigma}} - \sum_{n=1}^{\infty} \frac{2\mathrm{sin}(t\mathrm{ln}(2n^2))}{(2n^2)^{\sigma}} = 0 \qquad (3.4)$$

by Theorem 3.7. By multiplying equations (3.1) and (3.2) by $cos\phi$ and $sin\phi$, respectively, and subtracting, we get the equation:

$$(f_2(m) \equiv) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\mathrm{cos}(t\mathrm{ln}(mn))}{(mn)^{\sigma}} = 0$$

where we have used $\mathrm{cos}(\alpha+\beta) = \mathrm{cos}(\alpha)\mathrm{cos}(\beta) - \mathrm{sin}(\alpha)\mathrm{sin}(\beta)$. *Repeating the same procedure, we have the following equation.*

$$\sum_{n=1}^{\infty} \frac{\mathrm{cos}(t\mathrm{ln}(n^2))}{(n^2)^{\sigma}} - \sum_{n=1}^{\infty} \frac{2\mathrm{cos}(t\mathrm{ln}(2n^2))}{(2n^2)^{\sigma}} = 0. \qquad (3.5)$$

By defining A and B as

$$A = \sum_{n=1}^{\infty} \frac{\mathrm{sin}(t\mathrm{ln}(n^2))}{(n^2)^{\sigma}},$$

$$B = \sum_{n=1}^{\infty} \frac{\mathrm{cos}(t\mathrm{ln}(n^2))}{(n^2)^{\sigma}},$$

equations (3.4) and (3.5) are transformed to simpler forms:

$$A - 2^{1-\sigma}(A\cos(t\ln 2) + B\sin(t\ln 2)) = 0,$$
$$B - 2^{1-\sigma}(B\cos(t\ln 2) - A\sin(t\ln 2)) = 0,$$
$$\text{where } \sin(t\ln(2n^2)) = \sin(t\ln(n^2))\cos(t\ln 2) + \cos(t\ln(n^2))\sin(t\ln 2)$$
$$\text{and } \cos(t\ln(2n^2)) = \cos(t\ln(n^2))\cos(t\ln 2) - \sin(t\ln(n^2))\sin(t\ln 2) \text{ are used.}$$

These are the linear equations of A and B, and we define p, q, r and s so that these equations can look like the equations in Lemma 3.8

$$p = 1 - 2^{1-\sigma}\cos(t\ln 2),$$
$$q = -2^{1-\sigma}\sin(t\ln 2),$$
$$r = 2^{1-\sigma}\sin(t\ln 2),$$
$$s = 1 - 2^{1-\sigma}\cos(t\ln 2).$$

These relation has a property that when p=0 then s=0 and when q=0 then r=0. Therefore we can divide all cases into below four cases.
1) p=s=0, q=-r≠0
2) p=s≠0, q=-r=0
3) p=s≠0, q=-r≠0
4) p=s=0, q=-r=0
But, if 4) occurs then sin(tln2)=0 and therfore p=s=$1 - 2^{1-\sigma}$ *or p=s=*$1 + 2^{1-\sigma}$*. But, since p=s=0 there is a contradiction to the fact that* $1/2 < \sigma < 1$. *Since* $2^{1-\sigma}$ *can not be 1 or -1. Therefore, we only have to think about 3 cases mentioned as 1), 2) and 3).*

1) p=s=0, q=-r≠0
From the assumption the above linear equation change as follows:

$$pA + qB = 0, rA + sB = 0.$$
$$\textit{to}$$
$$qB = 0, rA = 0.$$

Since q=-r≠0 it results as A=B=0.

2) p=s≠0, q=-r=0
From the assumption the above linear equation change as follows:

$$pA + qB = 0, rA + sB = 0.$$
$$\textit{to}$$
$$pA = 0, sB = 0.$$

Since p=s≠0 it results as A=B=0.

3) p=s≠0, q=-r≠0

From the assumption all of p, q, r and s is non-zero values and satisfy the nessisary condition as follow

$$\frac{q}{p} = \frac{-2^{1-\sigma}\sin(t\ln 2)}{1-2^{1-\sigma}\cos(t\ln 2)}$$
$$\neq \frac{1-2^{1-\sigma}\cos(t\ln 2)}{2^{1-\sigma}\sin(t\ln 2)} = \frac{s}{r}$$

because to satisfy equality

$$\begin{aligned} (1-2^{1-\sigma}\cos(t\ln 2))^2 &= -(2^{1-\sigma}\sin(t\ln 2))^2 \\ \iff 1-2*2^{1-\sigma}\cos(t\ln 2)+(2^{1-\sigma}\cos(t\ln 2))^2 &= -(2^{1-\sigma}\sin(t\ln 2))^2 \\ \iff 1+(2^{1-\sigma}\cos(t\ln 2))^2+(2^{1-\sigma}\sin(t\ln 2))^2 &= 2*2^{1-\sigma}\cos(t\ln 2) \\ \iff 1+2^{2-2\sigma} &= 2^{2-\sigma}\cos(t\ln 2) \leq 2^{2-\sigma} \end{aligned}$$

but

$$2^{2-\sigma} \leq 1+2^{2-2\sigma}\ (AM-GM)$$

therefore, the only chance to satisfy equality is to $1 = 2^{2-2\sigma}$, *but this is impossible since* $1/2 < \sigma < 1$ *thus A=B=0 by Lemma 3.8.*

Since A=B=0 in any case, we have

$$\sum_{n=1}^{\infty}\frac{\sin(t\ln(n^2))}{(n^2)^\sigma} = \sum_{n=1}^{\infty}\frac{\cos(t\ln(n^2))}{(n^2)^\sigma} = 0.$$

From these, we have the following result for the Riemann zeta function $\zeta(2s)$*:*

$$\begin{aligned} 0 &= 0-0j \\ = \sum_{n=1}^{\infty}\frac{\cos(tln(n^2))-j\sin(t\ln(n^2))}{(n^2)^\sigma} &= \sum_{n=1}^{\infty}\frac{e^{-t\ln(n^2)j}}{(n^2)^\sigma} = \sum_{n=1}^{\infty}\frac{e^{-2t\ln(n)j}}{(n)^{2\sigma}} = \sum_{n=1}^{\infty}\frac{n^{-2tj}}{(n)^{2\sigma}} \\ &= \sum_{n=1}^{\infty}\frac{1}{(n)^{2\sigma+2jt}} = \sum_{n=1}^{\infty}\frac{1}{(n)^{2s}} = \zeta(2s) = 0. \end{aligned} \tag{3.6}$$

Above statement suggest that 2s is also a zero of the Riemann zeta function because of our assumption Re(2s) > 1 and the defintion of the Riemann zeta function for Re(t) > 1:

$$\zeta(t) = \sum_{n=1}^{\infty}\frac{1}{n^t}\quad (Re(t) > 1).$$

However, we already know by Theorem 2.3 that there is no zero for the region where Re(t) is bigger than 1. Therefore, the result (3.6) obtained by assuming that the Riemann zeta function has zero whose real value is not equal to 1/2, in the region where the real value of s is between 0 and 1, yields a contradiction. Now we can propose that the assumption we made was wrong and so that its real value must be 1/2 if there exist non-trivial zeros of the Riemann zeta function in the region where the real value of the complex variable is between 0 and 1. Therefore, the theorem is proved.

4 Conclusion

Following [1], we review a well-known proof of the Riemann hypothesis in the region where the real value of complex variable $Re(s) \leq 0$ *and* $Re(s) \geq 1$*. By using Theorem 3.7, 10 and Lemma 3.8, we prove Theorem 3.11 which is the main part of the Riemann hypothesis.*

5 Bibliography

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